Series 6

Task 2:

Let the Kalman decomposition be

$$V^*AV = \begin{array}{cc} \left[\begin{array}{c|c} A_1 & A_2 \\ \hline & A_3 \end{array}\right] & \begin{array}{c} r \\ n-r \end{array} \\ \begin{array}{cc} r & n-r \end{array} & \end{array} \qquad V^*B = \left[\begin{array}{c} B_1 \\ \hline 0 \end{array}\right] \begin{array}{c} r \\ n-r \end{array}$$

with $\operatorname{rank} K(A_1, B_1) = r$.

Then

$$\mathcal{Z}([\lambda I - A, -B]) = \mathcal{Z}\left(V^*[\lambda I - A, -B]\begin{bmatrix} V & 0 \\ 0 & I \end{bmatrix}\right)$$

$$= \mathcal{Z}\left(\left[\lambda\begin{bmatrix} I & \\ & I \end{bmatrix} - \begin{bmatrix} A_1 & A_2 \\ & A_3 \end{bmatrix}, \begin{bmatrix} -B_1 \\ 0 \end{bmatrix}\right]\right)$$

$$= \mathcal{Z}\left(\begin{bmatrix} \lambda I - A_1 & -A_2 & -B_1 \\ 0 & \lambda I - A_3 & 0 \end{bmatrix}\right)$$

$$= \mathcal{Z}\left(\left[\begin{array}{cc|c} \lambda I - A_1 & -B_1 & -A_2 \\ \hline \multicolumn{2}{c|}{0} & \lambda I - A_3 \end{array}\right]\right)$$

$$= \underbrace{\mathcal{Z}([\lambda I - A_1, -B_1])}_{= \emptyset \text{ Corollary 2.15}} \cup \mathcal{Z}(\lambda I - A_3)$$

$$= \sigma(A_3)$$

Spectrum of A_3, i.e., the set of eigenvalues

Task 3:

Let

(*) $$X(\lambda F + G)Y = \begin{bmatrix} \lambda F_1 + G_1 & * & * \\ 0 & \lambda F_2 + G_2 & * \\ 0 & 0 & \lambda F_3 + G_3 \end{bmatrix}$$

with $\lambda F_1 + G_1$ left prime and $\lambda F_3 + G_3$ right prime and F_2 invertible.

Then

$$\mathfrak{Z}(\lambda F + G) = \mathfrak{Z}(\lambda F_1 + G_1) \cup \mathfrak{Z}(\lambda F_2 + G_2) \cup \mathfrak{Z}(\lambda F_3 + G_3)$$

$$\overset{\text{Theorem } 1.13}{=} \emptyset \cup \mathfrak{Z}(\lambda F_2 + G_2) \cup \emptyset$$

$$= \sigma(-F_2^{-1}G_2).$$

Thus $\mathfrak{L}(\lambda F + G)$ is controllable if $\sigma(-F_2^{-1}G_2) = \emptyset$ (which is the case if and only if the block $\lambda F_2 + G_2$ vanishes).

Furthermore, $\mathfrak{L}(\lambda F + G)$ is stabilizable if and only if ~~$\mathfrak{Z}(\lambda F + G) \subset \mathbb{C}_-$~~

$$\sigma(-F_2^{-1}G_2) = \mathfrak{Z}(\lambda F + G) \subseteq \mathbb{C}_-.$$

~~For observability and reconstructability the same applies but one has to apply the staircase algorithm to the appropiate matrix polynomial.~~

For observability one has to check if a polynomial of the form

$$M(\lambda) = \lambda M_1 + M_0$$

~~has~~ is right prime (~~~~ if the polynomial is of higher order, compute the canonical linearization and check if it is right prime, cf. Lemma 4 on handout "First order systems)

Thus one computes (*) for

$$\lambda F + G = \lambda M_1 + M_0 .$$

If then $\lambda F_1 + G_1$ and $\lambda F_2 + G_2$ vanish, M is right prime, otherwise not.
(have zero column)

For reconstructability $\lambda F_2 + G_2$ may have more than zero colums but all eigenvalus of

$$(-F_2^{-1} G_2)$$

have to be in the left half plane.

Task 4:

By Theorem 1.13 let V be such that $[U, V]$ is unimodular. Partition the (unimodular) inverse as $\begin{array}{c} \begin{bmatrix} Q \\ P \end{bmatrix} \begin{array}{l} m \\ q-m \end{array} \end{array}$ such that

$$\begin{bmatrix} Q \\ P \end{bmatrix} [U, V] = \begin{bmatrix} I_m & 0 \\ 0 & I_{q-m} \end{bmatrix}$$

where P is left prime (again by Theorem 1.13). Then we see that

$$PU = 0 \qquad , \; PV = I_{q-m}$$

implies that U and V are right prime polynomial kernel and co-kernel spanning matrices of P.

Since by Theorem 1.13 $\mathfrak{J}(P) = \emptyset$ we see that $\mathfrak{B}(P)$ is controllable and by Theorem 2.13, 4.) we have

$$\mathfrak{B}(P) = \text{image}_{\ell_\infty}(U(\tfrac{d}{dx}))$$

Task 5:

1.) ⇒ 2.): Choose $(y_1, x_1) := (y, x)$ and
$$(y_2, x_2) := (0, 0).$$

2.) ⇒ 3.): Let (y_i, x_i, u_i), $i = 1, 2$ be with
$$\dot{x}_i = A(t)x_i + B(t)u_i \quad , \quad y_i = C(t)x_i + D(t)u_i,$$
and
$$y_1(t) = y_2(t) \quad , \quad u_1(t) = u_2(t) \qquad \forall t \in [t_0, t_1].$$
Let $x_+ \in \mathcal{C}_\infty^n$ be the unique solution of
$$\dot{x}_+ = A(t)x_+ + B(t)u_1 \quad , \quad x_+(t_0) = x_2(t_0).$$
Since $u_1(t) = u_2(t)$ $\forall t \in [t_0, t_1]$ we have
$$x_+(t) = x_2(t) \qquad \forall t \in [t_0, t_1]$$
Let $x_- \in \mathcal{C}_\infty^n$ be the unique solution of
$$\dot{x}_- = A(t)x_- + B(t)u_1 \quad , \quad x_-(t_1) = x_2(t_1).$$
As before this implies that $x_-(t) = x_2(t)$ in $[t_0, t_1]$.
Define
$$\tilde{x}_2(t) := \begin{cases} x_+(t) & , \ t \geq t_1 \\ x_2(t) = x_+(t) = x_-(t) & , \ t \in [t_0, t_1] \\ x_-(t) & , \ t \leq t_0 \end{cases} \in \mathcal{C}_\infty^n$$
and with this
$$\tilde{y}_2(t) := C(t)\tilde{x}_2(t) + D(t)u_1(t)$$
so that
$$(\tilde{y}_2, \tilde{x}_2, u_1) \in \tilde{\mathcal{L}}_0.$$

$$\Rightarrow \quad (\underbrace{y_1 - \tilde{y}_2}_{=:y}, \underbrace{x_2 - \tilde{x}_2}_{=:x}, \underbrace{u_1 - u_1}_{=0}) \in \tilde{\mathcal{L}}$$

$\Rightarrow \quad (y, x) \in \mathcal{L}$ and for $t \in [t_0, t_1]$ we have

$$y(t) = y_1(t) - \tilde{y}_2(t) = y_1(t) - y_2(t) = 0.$$

By assumption this shows that

$$\cancel{x_1(t)} \quad x(t) = 0 \qquad \forall t \in [t_0, t_1]$$

$$\Rightarrow \quad 0 = x(t) = x_1(t) - \tilde{x}_2(t) = x_1(t) - x_2(t)$$

for all $t \in [t_0, t_1]$.

<u>3.) $\Rightarrow$ 1.)</u>: Choose $u_1 = u_2 = 0$.

For 2.) $\Leftrightarrow$ 4.) $\Leftrightarrow$ 5.) one has to set

$$A(t) := -A^*(t), \quad B(t) := C^*(t)$$

in Theorem 2.2.

<u>Task 6</u>:

$$[(A, C) \text{ observable}] \overset{\text{Def}}{\Leftrightarrow} [\overbrace{\cancel{\ }}^{\text{rank}} K_0(A, C)) = n] \Leftrightarrow$$

$$[\text{kernel } K_0(A, C) = \{0\}] \overset{\text{Thm 2.25}}{\Leftrightarrow}$$

$$[\text{kernel } W(\tau) = \{0\}] \Leftrightarrow [W(\tau) > 0]$$

$$\overset{\text{Thm 2.21}}{\Leftrightarrow} [\forall (y_1, x_1, u_1), (y_2, x_2, u_2) \in \tilde{\mathcal{L}} \ \ldots\ldots]$$

Task 7: By Theorem 2.30 we can check if ~~f~~ the following matrices have right prime:

1.) (I_R, I_L, V) from I?

Setting
$$M(\lambda) := \begin{bmatrix} 1 & 1 & 0 \\ -R & 0 & 1 \\ 0 & \lambda L & -1 \end{bmatrix}, \quad R(\lambda) := \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

we have that

$$\mathcal{Z}(M) = \mathcal{Z}\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & R & 1 \\ 0 & \lambda L & -1 \end{bmatrix}\right) = \mathcal{Z}\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & \lambda L+R & -1 \end{bmatrix}\right)$$

$$= \mathcal{Z}\left(\begin{bmatrix} 1 & 1 & 0 \\ 0 & \lambda L+R & -1 \\ 0 & 0 & 1 \end{bmatrix}\right) = \mathcal{Z}(1) \cup \mathcal{Z}(\lambda L+R) \cup \mathcal{Z}(1)$$

$$= \left\{-\frac{R}{L}\right\}$$

which by Theorem 1.13 shows that M is not right prime. $\Rightarrow$ not observable

2.) (I_R, I_L, I) from V?

With
$$M(\lambda) = \begin{bmatrix} 1 & 1 & 1 \\ -R & 0 & 0 \\ 0 & \lambda L & 0 \end{bmatrix}$$

$$\mathcal{Z}(M) = \mathcal{Z}\left(\begin{bmatrix} 0 & 0 & 1 \\ -R & 0 & 0 \\ 0 & \lambda L & 0 \end{bmatrix}\right) = \{0\}$$

$\Rightarrow$ not observable

3.) (I_R, I, V) from I_L ?

With
$$\mathcal{M}(\lambda) = \begin{bmatrix} 1 & 1 & 0 \\ -R & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

we have
$$\operatorname{rank}(\mathcal{M}(\lambda_0)) = 3 \quad \forall \lambda_0 \in \mathbb{C}$$
and thus observability.

Staircase - Algorithm:

1.)
$$\left(\begin{bmatrix} & & \\ & & \\ & L & \end{bmatrix}, \begin{bmatrix} 1 & 1 & \\ -R & & 1 \\ & & -1 \end{bmatrix} \right) \to \left(\begin{bmatrix} & L & \\ & & \\ & & \end{bmatrix}, \begin{bmatrix} & & -1 \\ -R & & 1 \\ 1 & 1 & \end{bmatrix} \right) \to \left(\begin{bmatrix} L & L & \\ & & \\ & & \end{bmatrix}, \begin{bmatrix} -R & & -1 \\ 0 & & 1 \\ 0 & 1 & \end{bmatrix} \right) \quad \text{done}$$

2.)
$$\left(\begin{bmatrix} & & \\ & & \\ & L & \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 \\ -R & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) \to \left(\begin{bmatrix} & L & \\ & & \\ & & \end{bmatrix}, \begin{bmatrix} & & \\ -R & & \\ 1 & 1 & 1 \end{bmatrix} \right) \to \left(\begin{bmatrix} & L & \\ & & \\ & & \end{bmatrix}, \begin{bmatrix} & & \\ & & -R \\ 1 & 1 & 1 \end{bmatrix} \right)$$

$$\to \left(\begin{bmatrix} L & L & \\ & & \\ & & \end{bmatrix}, \begin{bmatrix} & & \\ & & -R \\ & 1 & 1 \end{bmatrix} \right) \qquad \text{done}$$

3.)
$$\left(\begin{bmatrix} & & \\ & & \\ & & \end{bmatrix}, \begin{bmatrix} 1 & 1 & \\ -R & & 1 \\ & & -1 \end{bmatrix} \right) \quad \text{nothing to do}$$

Task 8:

a) With $R(\lambda) := \left[\begin{array}{c|c|c} \lambda^2 m_1 + \lambda(d_1+d_2) + (k_1+k_2) & -\lambda d_2 - k_2 & -1 \\ \hline -\lambda d_2 - k_2 & \lambda^2 m_2 + \lambda d_2 + k_2 & 0 \end{array}\right]$

the system equations read

$$R\left(\tfrac{d}{dt}\right)\begin{bmatrix} q_1 \\ q_2 \\ f \end{bmatrix} = 0.$$

b) $$M(\lambda) = \begin{bmatrix} \lambda^2 m_1 + \lambda(d_1+d_2) + (k_1+k_2) \\ -\lambda d_2 - k_2 \end{bmatrix}$$

has full column rank (over $\mathbb{C}(\lambda)$) and

$$\mathcal{Z}(M) = \mathcal{Z}\left(\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} M\right) = \mathcal{Z}\left(\begin{bmatrix} \lambda^2 m_1 + \lambda d_1 + k_1 \\ \lambda d_2 + k_2 \end{bmatrix}\right).$$

Thus $\mathcal{Z}(M) = \emptyset$ if and only if

$$\lambda^2 m_1 + \lambda d_1 + k_1 \quad \text{and} \quad \lambda d_2 + k_2$$

have no common roots. Using the formula from the bottom of ~~the~~ "Series 5, Task 13" this ~~is equivalent to~~ means that

$$-\frac{d_1}{2m_1} \pm \sqrt{\left(\frac{d_1}{2m_1}\right)^2 - \frac{k_1}{m_1}} \neq -\frac{k_2}{d_2}.$$

is equivalent to the observability of q_1 from (q_2, f).

For reconstructability the common roots of

$$\lambda^2 m_1 + \lambda d_1 + k_1 \quad \text{and} \quad \lambda d_2 + k_2$$

only have to be in the left half plane.

2c)

$$\mathcal{M}(\lambda) = \begin{bmatrix} -\lambda d_2 - k_2 \\ \lambda^2 m_2 + \lambda d_2 + k_2 \end{bmatrix}$$

has full column rank (over $\mathbb{C}(\lambda)$) and

$$\mathcal{Z}(\mathcal{M}) = \mathcal{Z}\left(\begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} \mathcal{M} \right) = \mathcal{Z}\left(\begin{bmatrix} \lambda d_2 + k_2 \\ \lambda^2 m_2 \end{bmatrix} \right)$$

$$= \mathcal{Z}(\lambda^2 m_2) \cap \mathcal{Z}(\lambda d_2 + k_2)$$

$$= \{0\} \cap \left\{-\frac{k_2}{d_2}\right\} = \emptyset$$

if ~~k_2~~ $k_2 > 0$.

Thus q_2 is always (if $k_2 > 0$) observable from (q_1, f), and thus also reconstructable.

3.) After some computations one obtains that

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k_1+k_2}{m_1} & +\frac{k_2}{m_1} & -\frac{d_1+d_2}{m_1} & +\frac{d_2}{m_1} \\ +\frac{k_2}{m_2} & -\frac{k_2}{m_2} & +\frac{d_2}{m_2} & -\frac{d_2}{m_2} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_1} \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

→ to verify 2.) since in this case one asks if q_2 can be observed from the observation $y = q_1$ (and also f).

$$C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

→ to verify 1.) since in this case one asks if q_1 can be obtained from the observation $y = q_2$ (and $u = f$).

Then input (A^*, C_i^*) into the staircase algorithm.