Series 10:

Task 1:

1.) => 2.)

Let $v \in \mathbb{C}^n$ be with $Pv = b$. Then for $s \in \mathbb{R}$ we have that

$$z_s := \hat{z} + s\underbrace{(v - \hat{z})}_{=:y}$$

fulfills

$$Pz_s = P\hat{z} + s(\underbrace{Pv}_{=b} - \underbrace{P\hat{z}}_{=b}) = b$$

while

$$Py = Pv - P\hat{z} = b - b = 0.$$

Thus $y \in \text{kernel}\, P$ and by non-negativity

$$c := y^* H y \geq 0.$$

Define $\phi_v : \mathbb{R} \to \mathbb{R}$ through

$$\phi_v(s) := z_s^* H z_s = \hat{z}^* H \hat{z} + s2\,\mathrm{Re}\{y^* \underbrace{H\hat{z}}_{=-P^*\hat{\mu}}\} + (y^* H y)s^2$$

$$= \hat{z}^* H \hat{z} + 2s\,\mathrm{Re}\{-\underbrace{y^* P^*}_{=(Py)^*=0}\hat{\mu}\} + c \cdot s^2$$

$$= \hat{z}^* H \hat{z} + \underbrace{c}_{\geq 0} s^2$$

which means that ϕ_v has a minimum at $s = 0$. Thus $z_s|_{s=0} = \hat{z}$ is „better" (with respect to the chosen functional) than $z_s|_{s=1} = v$.

Since v was arbitrary this implies 2.)

<u>2.) => 1.)</u>

To show non-negativity let $y \in \mathbb{C}^n$ be with $Py = 0$. Set

$$z_s := \hat{z} + sy \qquad \text{for } s \in \mathbb{R}$$

so that

$$Pz_s = \underbrace{P\hat{z}}_{=b} + s\underbrace{Py}_{=0} = b$$

and thus (since $\hat{z}$ is "best" w.r.t. H)

$$\hat{z}^* H \hat{z} \le z_s^* H z_s = \hat{z}^* H \hat{z} + s\underbrace{2\operatorname{Re}\{y^* H \hat{z}\}}_{=:I_1} + s^2 \underbrace{y^* H y}_{=:I_2}$$

$$\Rightarrow \quad 0 \le s I_1 + s^2 I_2 \qquad \forall s \in \mathbb{R}$$

$$\Rightarrow \quad 0 \le I_2 = y^* H y \qquad \Rightarrow \text{non-negativity}$$

$$\Rightarrow 0 = I_1 \quad \Rightarrow \quad 0 = \operatorname{Re}\{y^* H \hat{z}\} \quad \forall y \text{ with } Py = 0$$

$\Rightarrow$ also $iy \in \operatorname{kernel}(P)$

$$\Rightarrow \quad 0 = \operatorname{Re}\{(iy)^* H \hat{z}\} = \dots = \operatorname{Im}\{y^* H \hat{z}\}$$

$$\Rightarrow \quad 0 = y^* H \hat{z} \qquad \forall y \text{ with } Py = 0$$

Now, let U, V be matrices which span the kernel and co-kernel of P ~~such that~~, i.e., with $r := \mathrm{rank}(P)$ let $U \in \mathbb{C}^{q,q-r}$, $V \in \mathbb{C}^{q,r}$ be with

- $PU = 0$
- $\mathrm{rank}(PV) = r$
- $[U, V]$ is invertible

(Existence: use echelon form, or SVD).

Then, for all $\alpha \in \mathbb{C}^{q-r}$ we have

$0 = P\underbrace{U\alpha}_{=y}$ and thus

$$0 = (U\alpha)^* H\hat{z} = \alpha^* U^* H\hat{z} \qquad \forall \alpha$$

$$\Rightarrow 0 = U^* H\hat{z} \quad \Rightarrow \quad 0 = -U^* H\hat{z}.$$

~~We can conclude that~~ Since (PV) has full column rank there exists a $\hat{\mu} \in \mathbb{C}^p$ such that

$$(PV)^* \hat{\mu} = -V^* H\hat{z}.$$

$$\Rightarrow P^*\hat{\mu} = [U, V]^{-*}[U, V]^* P^* \hat{\mu}$$

$$= [U, V]^{-*} \begin{bmatrix} U^* P^* \hat{\mu} \\ V^* P^* \hat{\mu} \end{bmatrix} = [U, V]^{-*} \begin{bmatrix} 0 \\ (PV)^* \hat{\mu} \end{bmatrix}$$

$$= [U, V]^{-*} \begin{bmatrix} -U^* H\hat{z} \\ -V^* H\hat{z} \end{bmatrix} = -[U, V]^{-*}[U, V] H\hat{z}$$

$$= -H\hat{z}.$$

Task 2:

Since by S. 9, T. 2 we have

$$N^\sim(\lambda) = \sum_{i=0}^{k} \lambda^i (-1)^i N_i^*$$

we see that by equating coefficients in $N = N^\sim$ that

$$N_i = (-1)^i N_i^*$$

which for

$i = 2k$ even: $N_i = N_i^*$

$i = 2k+1$ odd: $N_i = (-1) N_i^*$

Task 3:

Hermitian:

$$(x^* M x)^* = x^* M^* x = x^* M x$$

$$\Rightarrow \overline{x^* M x} = x^* M x \in \mathbb{R}$$

skew-Hermitian:

$$\overline{x^* M x} = (x^* M x)^* = x^* M^* x = -x^* M x$$

$$\Rightarrow \operatorname{Re}(x^* M x) = \frac{1}{2}\left(x^* M x + \overline{x^* M x}\right)$$

$$= \frac{1}{2}\left(x^* M x - x^* M x\right) = 0$$

$$\Rightarrow x^* M x \in i\mathbb{R}$$

Task 4:

Write $f(t) = f_r(t) + i\, f_i(t)$, with $f_r, f_i : \mathbb{R} \to \mathbb{R}$. Then by definition

$$\int_{t_0}^{t_1} f(t)\,dt := \int_{t_0}^{t_1} f_r(t)\,dt + i \int_{t_0}^{t_1} f_i(t)\,dt$$

and thus

$$\mathrm{Re}\left\{\int_{t_0}^{t_1} f(t)\,dt\right\} = \int_{t_0}^{t_1} f_r(t)\,dt = \int_{t_0}^{t_1} \mathrm{Re}\{f(t)\}\,dt$$

Task 5:

By setting $H := \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}$ and using $z = \begin{bmatrix} x \\ u \end{bmatrix}$ we can use Theorem 4.5 to verify that

$$\inf_{\substack{z \in \mathfrak{B}_+(\lambda F + G) \\ z(t) = z_0(t),\ t \leq 0}} \int_0^\infty z^*(t) H z(t)\,dt \overset{\text{Thm 4.5}}{=} \inf_{\substack{z \in \mathfrak{B}_+(\lambda F + G) \\ \underbrace{Fz(0)}_{= [I, 0]\begin{bmatrix} x(0) \\ u(0) \end{bmatrix} = x(0)} = Fz_0(0)}} \int_0^\infty \ldots\,dt$$

$$= \inf_{\substack{(x,u) \in \mathfrak{B}_+(\lambda F + G) \\ x(0) = x_0\ (:= [I, 0] z_0(0))}} \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^* \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} dt$$

$$= \inf_{\ldots} \int_0^\infty x^*(t) Q x(t) + 2\,\mathrm{Re}\{x^*(t) S u(t)\} + u^*(t) R u(t)\,dt.$$

On the other hand the optimality system can be rewritten as

$$0 = \begin{bmatrix} 0 & F \\ -F^* & 0 \end{bmatrix} \begin{bmatrix} \mathring{\hat{\mu}}(t) \\ \mathring{\hat{z}}(t) \end{bmatrix} + \begin{bmatrix} 0 & G \\ G^* & H \end{bmatrix} \begin{bmatrix} \hat{\mu}(t) \\ \hat{z}(t) \end{bmatrix}$$

$$= \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathring{\hat{\mu}} \\ \mathring{\hat{x}} \\ \mathring{\hat{u}} \end{bmatrix} + \begin{bmatrix} 0 & -A & -B \\ -A^* & Q & S \\ -B^* & S^* & R \end{bmatrix} \begin{bmatrix} \hat{\mu} \\ \hat{x} \\ \hat{u} \end{bmatrix}$$

drop ^-symbols: $\hat{\mu} = \mu$, $\hat{z} = \ldots$

$$\iff \begin{bmatrix} 0 & I & 0 \\ -I & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathring{\mu} \\ \mathring{x} \\ \mathring{u} \end{bmatrix} = \begin{bmatrix} 0 & A & B \\ A^* & -Q & -S \\ B^* & -S^* & -R \end{bmatrix} \begin{bmatrix} \mu \\ x \\ u \end{bmatrix}$$

$$\iff \begin{bmatrix} I & & \\ & -I & \\ & & 0 \end{bmatrix} \begin{bmatrix} \mathring{x} \\ \mathring{\mu} \\ \mathring{u} \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ -Q & A^* & -S \\ -S^* & B^* & -R \end{bmatrix} \begin{bmatrix} x \\ \mu \\ u \end{bmatrix}$$

$$\iff \begin{bmatrix} I & & \\ & -I & \\ & & 0 \end{bmatrix} \begin{bmatrix} \mathring{x} \\ \mathring{\mu} \\ \mathring{u} \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ -Q & A^* & -S \\ S^* & -B^* & R \end{bmatrix} \begin{bmatrix} x \\ \mu \\ u \end{bmatrix}$$

$$\iff \begin{bmatrix} I & & \\ & I & \\ & & 0 \end{bmatrix} \begin{bmatrix} \mathring{x} \\ -\mathring{\mu} \\ \mathring{u} \end{bmatrix} = \begin{bmatrix} A & 0 & B \\ -Q & -A^* & -S \\ S^* & +B^* & R \end{bmatrix} \begin{bmatrix} x \\ -\mu \\ u \end{bmatrix}$$

which is the given optimality system by setting $\lambda := -\mu$.

From this it should be possible to give a mathematically correct proof with the help of Theorems 4.3 & 4.4.

Task 6: With $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}$ we have

$$[V^* J V = J] \iff [JV = VJ]$$

$$\iff \left[\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \right]$$

$$\iff \left[\begin{bmatrix} Q_{21} & Q_{22} \\ -Q_{11} & -Q_{12} \end{bmatrix} = \begin{bmatrix} -Q_{12} & Q_{11} \\ -Q_{22} & Q_{21} \end{bmatrix} \right]$$

$$\iff \left[Q_{11} = Q_{22} \quad \wedge \quad Q_{21} = -Q_{12} \right]$$

Task 7: Since $P^* = I^* - 2\left(\frac{uu^*}{\|u\|_2^2}\right)^* = P$
we have

$$P^* P = I - 4\frac{uu^*}{u^*u} + 4\frac{uu^*uu^*}{(u^*u)^2}$$

$$= I - 4\frac{uu^*}{u^*u} + 4\frac{(uu^*)\, uu^*}{(uu^*)^2} = I$$

and with

$$\|u\|_2^2 = \|x\|^2 + e^{i\omega}\|x\| \underbrace{x^* e_p}_{= \overline{x_p} = r e^{-i\omega}} + e^{-i\omega}\|x\| e_p^* x + \|x\|^2$$

$$= 2\|x\|^2 + 2\|x\| r = 2\|x\| (\|x\| + r)$$

we obtain

$$P e_p = e_p - \frac{2}{\|u\|^2}\left(u \left(x^* + e^{-i\omega}\|x\| e_p^* \right) e_p \right)$$

$$= e_p - \frac{2}{\|u\|^2}\left(u\left(\underbrace{\overline{x_p}}_{=re^{-i\omega}} + e^{-i\omega}\|x\|\right)\right)$$

$$= e_p - \frac{2}{\|u\|^2}\left(u\, e^{-i\omega}\left(r + \|x\|\right)\right)$$

$$= e_p - \frac{2}{2\|x\|\cancel{(\|x\|+r)}}\left(u\, e^{-i\omega}\cancel{(r+\|x\|)}\right)$$

$$= e_p - \frac{1}{\|x\|}\left(x + e^{i\omega}\|x\|\, e_p\right)e^{-i\omega}$$

$$= e_p - \frac{1}{\|x\|}e^{-i\omega}x - e_p = -\left(\|x\|\, e^{i\omega}\right)^{-1}x$$

$$\Rightarrow e_p = \underbrace{P^*}_{=P} P e_p = -P\left(\|x\| e^{i\omega}\right)^{-1}x$$

$$\Rightarrow Px = \left(-\|x\|\, e^{i\omega}\right)e_p .$$

Task 8:

(a) We have

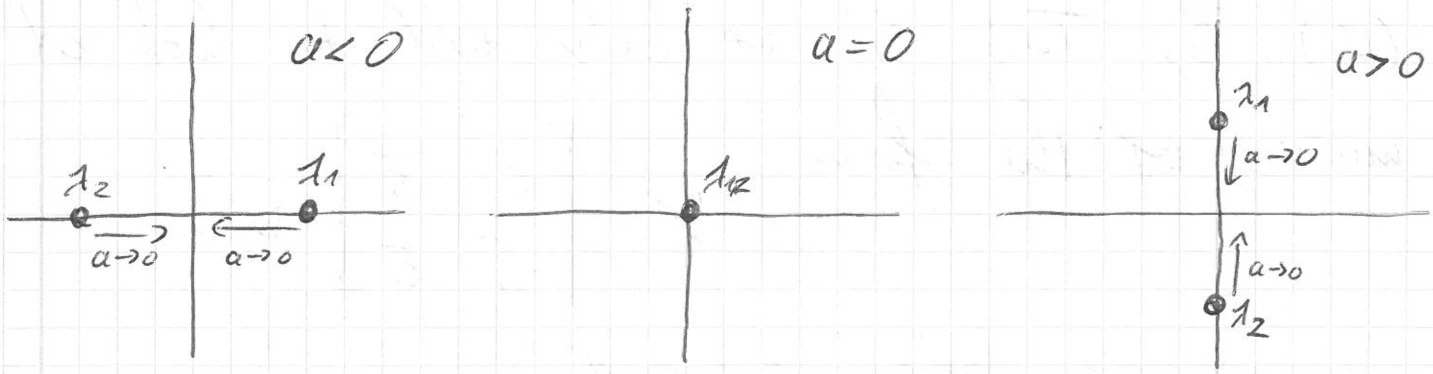
$$\mathcal{Z}\left(\lambda\left(\begin{bmatrix} 1 & \lambda \\ -\lambda & a \end{bmatrix}\right)\right) = \mathcal{Z}\left(\begin{bmatrix} 1 & \lambda \\ 0 & a+\lambda^2 \end{bmatrix}\right) = \mathcal{Z}(\lambda^2 + a)$$

$$= \{\pm\sqrt{-a}\}$$

and for $\lambda_1 := \sqrt{-a}$ we have

$$N(\lambda_1)x_1 = \begin{bmatrix} 1 & \sqrt{-a} \\ -\sqrt{-a} & a \end{bmatrix}\begin{bmatrix} x_{1,1} \\ x_{1,2} \end{bmatrix} \overset{!}{=} 0$$

$$\Rightarrow x_0 = \begin{bmatrix} -\sqrt{-a} \\ 1 \end{bmatrix}$$



(b) For $a < 0$ we have that

$$\lambda_1 = \sqrt{-a} \in \mathbb{R} \quad , \quad Re(\lambda_1) > 0$$

and thus we can use the construction of Lemma 4.6 with $\alpha = -\sqrt{-a}$, $\tilde{w}_p = 1$

$$\Rightarrow \left(\frac{\tilde{w}_p}{\alpha}\right) = -\frac{1}{\sqrt{-a}}$$

$$\Rightarrow S := \frac{1}{\sqrt{1 + \left(-\frac{1}{\sqrt{-a}}\right)^2}} = \frac{1}{\sqrt{1 + \frac{1}{-a}}} = \sqrt{\frac{-a}{-a+1}}$$

$$\Rightarrow C := -\frac{1}{\sqrt{-a}} \cdot \sqrt{\frac{-a}{-a+1}} = -\frac{1}{\sqrt{-a+1}}$$

so that

$$V := \hat{G} := \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \quad \text{is unitary}$$

with

$$V x_0 = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} -\sqrt{-a} \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{-a}{-a+1}} - \sqrt{\frac{-a}{-a+1}} \\ -\frac{-a}{\sqrt{-a+1}} - \frac{1}{\sqrt{-a+1}} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \frac{a-1}{\sqrt{-a+1}} \end{bmatrix} = \begin{bmatrix} 0 \\ -\sqrt{-a+1} \end{bmatrix}$$

(c) By Task 6 we only need to look at matrices of the form $V = \begin{bmatrix} \bar{c} & \bar{s} \\ -\bar{s} & \bar{c} \end{bmatrix}$ with

<u>complex</u> numbers $c, s \in \mathbb{C}$.

Assume to the contrary that one can choose c, s such that

$$\begin{bmatrix} 0 \\ * \end{bmatrix} = \underbrace{\begin{bmatrix} c & -s \\ s & c \end{bmatrix}}_{=V^*} \begin{bmatrix} -\sqrt{-a} \\ 1 \end{bmatrix} = \begin{bmatrix} -c\sqrt{-a} - s \\ * \end{bmatrix}$$

$$\Rightarrow \quad s = -c\sqrt{-a} \underset{a>0}{\overset{\wedge}{=}} -ic\sqrt{a}.$$

Then we would have

$$I = V^*V = \begin{bmatrix} c & ic\sqrt{a} \\ -ic\sqrt{a} & c \end{bmatrix} \begin{bmatrix} \bar{c} & i\bar{c}\sqrt{a} \\ -i\bar{c}\sqrt{a} & \bar{c} \end{bmatrix}$$

$$= \begin{bmatrix} |c|^2 + |c|^2 a & |c|^2 i\sqrt{a} + i|c|^2\sqrt{a} \\ * & |c|^2 a + |c|^2 \end{bmatrix}$$

(1,2)-entry
$$\Rightarrow \quad 0 = 2i\underbrace{\sqrt{a}}_{>0}|c|^2 \quad \Rightarrow \quad |c|^2 = 0 \quad \Rightarrow c = 0$$

(1,1)-entry
$$\Rightarrow \quad 1 = |c|^2 + |c|^2 a = 0 \quad ↯.$$

(d) By Task 6, again, we only consider

$$V = \begin{bmatrix} \bar{c} & \bar{s} \\ -\bar{s} & \bar{c} \end{bmatrix}, \quad c, s \in \mathbb{C}$$

and assume to the contrary that

$$\begin{bmatrix} m_1 & m_2 \\ \overline{m_2} & 0 \end{bmatrix} = V^* \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix} V = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} \bar{c} & \bar{s} \\ -a\bar{s} & a\bar{c} \end{bmatrix}$$

$$= \begin{bmatrix} * & * \\ * & |s|^2 + a|c|^2 \end{bmatrix}$$

$$\Rightarrow \quad 0 = |s|^2 + a|c|^2 \overset{a>0}{\Rightarrow} |s|^2 = 0, \ |c|^2 = 0$$

$$\Rightarrow \quad s = c = 0 \quad \Rightarrow \quad V = 0$$ which contradicts the assumption that V is unitary.

Task 9:

1.) Let $N(\lambda) = \lambda \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + M$, $M \in \mathbb{C}^{2,2}$,

let $\lambda_0 \in \sigma(N)$ with $\mathrm{Re}(\lambda_0) < 0$, and let $x_0 \in \mathbb{C}^2 \setminus \{0\}$ be with

$$N(\lambda_0)\, x_0 = 0.$$

By Lemma 4.6 choose $V \in \mathbb{C}^{2,2}$ unitary with $V^* \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} V = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ such that

$$V^* x_0 = \begin{bmatrix} 0 \\ \beta \end{bmatrix} = \beta e_2 \qquad , \beta \neq 0 .$$

$$\Rightarrow \quad 0 = V^* N(\lambda_0) x_0 \quad = \underbrace{V^* M(\lambda_0) V}_{=: \tilde{M}(\lambda_0)} \underbrace{V^* x_0}$$

$$= \left(\lambda_0 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \underbrace{V^* M V}_{=: \begin{bmatrix} m_{11} & m_{12} \\ \overline{m_{12}} & m_{22} \end{bmatrix}} \right) \begin{bmatrix} 0 \\ \beta \end{bmatrix} \qquad , m_{11}, m_{22} \in \mathbb{R}, \; m_{12} \in \mathbb{C}$$

$$= \begin{bmatrix} m_{11} & \lambda_0 + m_{12} \\ -\lambda_0 + \overline{m_{12}} & m_{22} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \beta$$

$$\overset{\beta \neq 0}{\Rightarrow} \quad \begin{bmatrix} \lambda_0 + m_{12} \\ m_{22} \end{bmatrix} = 0 \qquad \Rightarrow \; m_{22} = 0 \; , m_{12} = -\lambda_0$$

$$\Rightarrow \; \tilde{N}(\lambda) \; = \; \lambda \left[\begin{array}{c|c} 0 & 1 \\ \hline -1 & 0 \end{array}\right] + \left[\begin{array}{c|c} m_{11} & -\lambda_0 \\ \hline -\lambda_0 & 0 \end{array}\right]$$

2.)

$$\mathfrak{Z}\left(\begin{bmatrix} M_{11} & \times & \lambda I - M_{12} & 0 \\ \times & \times & \times & \lambda - \lambda_0 \\ \lambda I - M_{12}^* & \times & M_{22} & 0 \\ 0 & -\lambda - \bar{\lambda}_0 & 0 & 0 \end{bmatrix}\right)$$

$$= \mathfrak{Z}\left(\begin{bmatrix} 0 & M_{11} & \lambda I - M_{12} & \times \\ \lambda - \lambda_0 & \times & \times & \times \\ 0 & \lambda I - M_{12}^* & M_{22} & \times \\ 0 & 0 & 0 & -\lambda - \bar{\lambda}_0 \end{bmatrix}\right)$$

$$= \mathfrak{Z}\left(\begin{bmatrix} \lambda - \lambda_0 & \times & \times & \times \\ 0 & M_{11} & \lambda I - M_{12} & \times \\ 0 & -\lambda I - M_{12}^* & M_{22} & \times \\ 0 & 0 & 0 & -\lambda - \bar{\lambda}_0 \end{bmatrix}\right)$$

$$= \underbrace{\mathfrak{Z}(\lambda - \lambda_0)}_{=\{\lambda_0\}} \cup \mathfrak{Z}\left(\begin{bmatrix} M_{11} & \lambda I - M_{12} \\ -\lambda I - M_{12}^* & M_{22} \end{bmatrix}\right) \cup \underbrace{\mathfrak{Z}(-\lambda - \bar{\lambda}_0)}_{=\{-\bar{\lambda}_0\}}$$