Series 11:

Task 1:

We have (with $P(\lambda) = \lambda F + G$)

$$\operatorname{rank}_{\mathbb{C}(\lambda)}(\mathcal{H}) = \operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} O & P \\ P^{\sim} & H \end{bmatrix}$$

$$= \operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} I & -PH^{-1} \\ O & I \end{bmatrix} \begin{bmatrix} O & P \\ P^{\sim} & H \end{bmatrix}$$

$$= \operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} -PH^{-1}P^{\sim} & O \\ P^{\sim} & H \end{bmatrix}$$

$$= \underbrace{\operatorname{rank}_{\mathbb{C}(\lambda)}(PH^{-1}P^{\sim})}_{\substack{=p \\ \text{shown in the following}}} + \underbrace{\operatorname{rank}_{\cancel{\mathbb{C}(\lambda)}} H}_{\substack{=q \\ \text{by Assumption 4.8}}}$$

P has full row rank, since F has full row rank. Thus there exists a $\lambda_0 \in \mathbb{C}$ such that $\operatorname{rank} P(\lambda_0) = p = \operatorname{rank} P^{\sim}(\lambda_0)$.

$$\Rightarrow \operatorname{rank} P(\lambda_0) H^{-1} P^{\sim}(\lambda_0) = p$$

$$\Rightarrow p \geq \operatorname{rank}_{\mathbb{C}(\lambda)} PH^{-1}P^{\sim} \overset{\text{Lemma 4.9}}{\geq} \operatorname{rank} P(\lambda_0)H^{-1}P^{\sim}(\lambda_0) = p.$$

Task 2:

Compute the singular value decomposition of

$$Y = U \Sigma V^* \quad \text{and}$$

set $\quad W := VU^* \quad$ to obtain

$$YW = (U\Sigma V^*)(VU^*) = U\underbrace{\Sigma}_{\text{diagonal and real}}U^* .$$

<u>Task 3:</u>

$$\begin{bmatrix} I & 0 \\ Z^* & I \end{bmatrix}\begin{bmatrix} 0 & \lambda F+G \\ -\lambda F^*+G^* & H \end{bmatrix}\begin{bmatrix} I & Z \\ & I \end{bmatrix} = \begin{bmatrix} 0 & \lambda F+G \\ -\lambda F^*+G^* & C^*C \end{bmatrix}$$

$$\|$$

$$\begin{bmatrix} I & \\ Z^* & I \end{bmatrix}\begin{bmatrix} 0 & \lambda F+G \\ -\lambda F^*+G^* & -\lambda F^*Z+G^*Z+H \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \lambda F+G \\ -\lambda F^*+G^* & \lambda Z^*F+Z^*G-\lambda F^*Z+G^*Z+H \end{bmatrix}$$

$$\Leftrightarrow \quad C^*C = \lambda(Z^*F - F^*Z) + H + G^*Z + Z^*G$$

$$\Leftrightarrow \quad Z^*F - F^*Z = 0 \quad \text{and}$$

$$H + G^*Z + Z^*G = C^*C$$

Task 4:

$$\left(N(\imath\omega_0)\right)^* = N^*(\underbrace{\overline{\imath\omega_0}}_{=-\imath\,\omega_0 = -\imath\omega_0}) = N^*(-\overline{\imath\omega_0})$$

$$= N^{\sim}(\imath\omega_0) = N(\imath\omega_0)$$

Task 5: We have

$$\Theta(z(t_1)) - \Theta(z(t_0)) = \int_{t_0}^{t_1} \frac{d}{dt}\left(\Theta(z(t))\right) dt$$

$$= \int_{t_0}^{t_1} \frac{d}{dt}\left(z^*(t)\, F^* Z z(t)\right) dt$$

$$= \int_{t_0}^{t_1} \underbrace{\dot{z}^*(t) F^*}_{= (F\dot{z}(t))^* \underset{z \in \mathfrak{B}(\imath F + G)}{=} (-Gz(t))^*} Z z(t) + z^*(t) Z^* F \dot{z}(t)\, dt$$

$$= \int_{t_0}^{t_1} - z^*(t) G^* Z z(t) - z^*(t) Z^* G z(t)\, dt$$

$$= -\int_{t_0}^{t_1} z^*(t) \underbrace{\left[G^* Z + Z^* G\right]}_{\underset{\text{Task 3}}{=} C^*C - H} z(t)\, dt$$

$$= \int_{t_0}^{t_1} z^*(t) H z(t)\, dt - \underbrace{\int_{t_0}^{t_1} \underbrace{z^*(t) C^* C z(t)}_{= \|Cz(t)\|_2^2 \geq 0} dt}_{\geq 0}$$

$$\leq \int_{t_0}^{t_1} z^*(t) H z(t)\, dt .$$

Furthermore, if $z \in \mathcal{L}_+(iF+G)$ is with $z(t)=0$, $t \leq 0$, then

$$\int_0^\infty z^*(t) H z(t)\,dt = \lim_{R\to\infty} \int_0^R z^*(t) H z(t)\,dt$$

$$\geq \lim_{R\to\infty} \theta(z(R)) - \theta(z(0))$$

$$= \theta\Big(\underbrace{\lim z(R)}_{=0\text{, since } z\in\mathcal{L}_+(\ldots)}\Big) - \theta\big(\underbrace{z(0)}_{=0\text{, since } z(t)=0,\, t\leq 0}\big)$$

$\theta(y) = y^* Z^* F y$ is continuous

$$= \theta(0) - \theta(0) = 0.$$

This implies non-negativity.

<u>Task 6:</u> If $\hat{z}$ is optimal by Theorem 4.4 there exists a $\hat{\mu} \in \mathcal{C}_+^p$ such that for $t \geq 0$

$$\underbrace{(\hat{\mu}(t), \hat{z}(t))}_{:= \begin{bmatrix} \hat{\mu}(t) \\ \hat{z}(t) \end{bmatrix}} \in \mathcal{L}_+\left(\begin{bmatrix} 0 & P \\ P^\sim & H \end{bmatrix}\right) \overset{\text{Theorem 4.10 2.) b)}}{=} \begin{bmatrix} Z \\ I \end{bmatrix} \mathcal{L}_+\left(\begin{bmatrix} P \\ C \end{bmatrix}\right)$$

$$\Rightarrow 0 = C\hat{z}(t)$$

$$\left(\text{and } \hat{\mu}(t) = Z\hat{z}(t)\right) \text{ for } t \geq 0.$$

Thus we obtain

$$\int_0^\infty \hat{z}^*(t) H \hat{z}(t)\,dt = \int_0^\infty \hat{z}^*\left(C^*C - Z^*G - G^*Z\right)\hat{z}\,dt$$

$$= \int_0^\infty \|\underbrace{C\hat{z}(t)}_{=0}\|_2^2\,dt - \int_0^\infty \underbrace{\hat{z}^*G^*}_{=(G\hat{z})^*=(-F\dot{\hat{z}})^*} Z\hat{z} + \hat{z}^*Z^*Gz\,dt$$

$$= \int_0^\infty \dot{\hat{z}}^*F^*Z\hat{z} + \hat{z}^*Z^*F\dot{\hat{z}}\,dt$$

$$= \int_0^\infty \frac{d}{dt}\left(\hat{z}^*F^*Z\hat{z}\right)dt$$

$$= \underbrace{\left(\lim_{R\to\infty} \hat{z}^*(R)\,F^*Z\hat{z}(R)\right)}_{=0\text{, da } \hat{z}\in\mathcal{L}_+(\dots)} - \hat{z}^*(0)F^*Z\hat{z}(0)$$

$$= \Theta(\hat{z}(0)).$$

Task 7:

(a) 1.) $\Rightarrow$ 2.)

$$\mathcal{G}\mathcal{J} = (\mathcal{G}\mathcal{J})^* = \mathcal{J}^*\mathcal{G}^* = -\mathcal{J}\mathcal{G}$$

2.) $\Rightarrow$ 3.)

$$\mathcal{J}\mathcal{G} = \mathcal{J}(\mathcal{G}\mathcal{J})\mathcal{J}^{-1} = \underbrace{-\mathcal{J}\mathcal{J}}_{=I}\mathcal{G}^*\mathcal{J}^{-1} = \mathcal{G}^*\mathcal{J}^* = (\mathcal{J}\mathcal{G})^*$$

3.) $\Rightarrow$ 4.) Denote the Hermitian matrix $\mathcal{J}\mathcal{G}$ in the form

$$J\mathcal{H} =: \begin{bmatrix} +C & -B^* \\ -B & -A \end{bmatrix}, \text{ with } C = C^*, A = A^*.$$

Then

$$\mathcal{H} = J^{-1}J\mathcal{H} = \begin{bmatrix} & -I \\ I & \end{bmatrix}\begin{bmatrix} +C & -B^* \\ -B & -A \end{bmatrix}$$

$$= \begin{bmatrix} B & A \\ C & -B^* \end{bmatrix}.$$

4.) => 1.) We have that

$$J\mathcal{H} = \begin{bmatrix} & I \\ -I & \end{bmatrix}\begin{bmatrix} B & A \\ C & -B^* \end{bmatrix} = \begin{bmatrix} C & -B^* \\ -B^* & -A \end{bmatrix}$$

is Hermitian

(b) It is

$$\sigma(\mathcal{H}) = \mathcal{Z}(\lambda I - \mathcal{H}) = \mathcal{Z}((\lambda I - \mathcal{H})J)$$

$$\cancel{= \mathcal{Z}(J(\lambda I - \mathcal{H})) = \mathcal{Z}(\ldots}$$

$$= \mathcal{Z}(\underbrace{\lambda J}_{\text{Skew-Hermitian}} - \underbrace{(\mathcal{H}J)}_{\text{Hermitian}}) =: \mathcal{Z}(N(\lambda))$$

where $N^\sim = N$ is para-Hermitian. Thus $\mathcal{Z}(N) = \sigma(\mathcal{H})$ is symmetric w.r.t. the imaginary axis.

(c) Since $N(\lambda) := \lambda J - \mathcal{H} J$ is para-Hermitian without imaginary zeros

$$\mathcal{Z}(N) \cap (i\mathbb{R}) \overset{(b)}{=} \sigma(\mathcal{H}) \cap (i\mathbb{R}) \underset{\substack{\uparrow \\ \text{by Assumption}}}{=} \emptyset$$

there exists (by Theorem 4.7) a unitary $V = S \in \mathbb{C}^{2n \times 2n}$ such that $V^* J V = J$ (i.e., it is symplectic) and

$$V^* N(\lambda) V = \lambda J - \underbrace{V^* \mathcal{H} J V}_{= \begin{bmatrix} K & L \\ L^* & 0 \end{bmatrix}} = \left[\begin{array}{c|c} K & \lambda I - L \\ \hline -\lambda I - L^* & 0 \end{array}\right]$$

where $L = [\triangle]$ and

$$\mathbb{C}_- \supseteq \mathcal{Z}(\lambda I - L) = \sigma(L).$$

This implies $\begin{bmatrix} K & L \\ L^* & 0 \end{bmatrix} = V^* \mathcal{H} V \underbrace{V^* J V}_{= J} = V^* \mathcal{H} V J$

$$\Rightarrow V^* \mathcal{H} V = \begin{bmatrix} K & L \\ L^* & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}}_{= J^{-1} = -J} = \left[\begin{array}{c|c} L & -K \\ \hline 0 & -L^* \end{array}\right]$$

Task 8: With $P(\lambda) := \begin{bmatrix} \lambda & -1 \end{bmatrix}$

and $H = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}$

the matrix polynomial for the optimality system is

$$\mathcal{H}(\lambda) := \begin{bmatrix} 0 & P \\ P^\sim & H \end{bmatrix} = \begin{bmatrix} 0 & \lambda & -1 \\ -\lambda & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Thus we can choose $X_1 = I \in \mathbb{C}^{3,3}$

and

$$X_2^* \begin{bmatrix} 0 & \lambda & -1 \\ -\lambda & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{=: X_2}$$

should become zero

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & \lambda & -1 \\ -\lambda & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left[\begin{array}{cc|c} -1 & \lambda & \\ -\lambda & 1 & \\ \hline & & 1 \end{array}\right]$$

The zeros of $N_1(\lambda) := \lambda \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}}_{= \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix}}$

are then

$$\mathfrak{Z}(N_1) = \mathfrak{Z}\left(\begin{bmatrix} -\lambda & \lambda \\ -\lambda & 1 \end{bmatrix}\right) = \mathfrak{Z}\left(\underbrace{\begin{bmatrix} 1 & \\ -\lambda & 1 \end{bmatrix}}_{\text{unimodular}}\begin{bmatrix} -\lambda & \lambda \\ -\lambda & 1 \end{bmatrix}\right)$$

$$= \mathfrak{Z}\left(\begin{bmatrix} -\lambda & \lambda \\ 0 & -\lambda^2+1 \end{bmatrix}\right) = \mathfrak{Z}(-\lambda^2+1) = \{\pm 1\}$$

Choosing the stable zero $\lambda_1 = -1$ we have that (in the notation similar to Lemma 4.6)

$$N(\lambda_1)x_1 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{=: x_1} = 0$$

shall be determined

and thus we choose

$$\underbrace{\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_{=: V_3^*}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}\frac{1}{\sqrt{2}} = \begin{bmatrix} 0 \\ -\sqrt{2} \end{bmatrix}$$

and obtain the para-Hermitian Schur form of Theorem 4.7

$$V_3^* N_1(\lambda) V_3 = V_3^* \begin{bmatrix} -\lambda & \lambda \\ -\lambda & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \lambda-1 & \lambda+1 \\ -(\lambda-1) & \lambda+1 \end{bmatrix} = \frac{1}{2}\left[\begin{array}{c|c} 0 & 2(\lambda+1) \\ \hline -2(\lambda-1) & 0 \end{array}\right]$$

$$= \left[\begin{array}{c|c} 0 & \lambda+1 \\ \hline -\lambda+1 & 0 \end{array}\right].$$

all stable zeros are here

This means that (in the proof of Theorem 4.10)

$$X_3 := \begin{bmatrix} V_3 & \\ & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and thus

$$Y := X_1 X_2 X_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \\ & & 1 \end{bmatrix}$$

$$= \left[\begin{array}{c|cc} 1 & -1 & 0 \\ \hline 1 & 1 & 0 \\ 1 & -1 & \sqrt{2} \end{array}\right] \frac{1}{\sqrt{2}}$$

$$=: \left[\begin{array}{c|c} Y_1 & Y_2 \\ \hline Y_3 & Y_4 \end{array}\right]$$

$$\Rightarrow \quad Y_4^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \sqrt{2} = \begin{bmatrix} \sqrt{2} & 0 \\ 1 & 1 \end{bmatrix}$$

$$\Rightarrow \quad Y_4^{-1} Y_3 = \begin{bmatrix} \sqrt{2} & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} \sqrt{2} \\ 2 \end{bmatrix} \frac{1}{\sqrt{2}} = \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix}$$

$$\Rightarrow \quad Y_2 Y_4^{-1} Y_3 = \frac{1}{\sqrt{2}} [-1, 0] \begin{bmatrix} 1 \\ \sqrt{2} \end{bmatrix} = -\frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad Y_1 - Y_2 Y_4^{-1} Y_3 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

which means that

$$\hat{Y} := \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -1 & \sqrt{2} & 0 \\ -\sqrt{2} & 1 & 1 \end{array}\right] \left[\begin{array}{c|cc} \frac{1}{\sqrt{2}} & & \\ \hline & 1 & \\ & & 1 \end{array}\right]$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \sqrt{2} & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

and

$$Y\hat{Y} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & -1 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \sqrt{2} & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{2}{\sqrt{2}} & -\sqrt{2} & 0 \\ 0 & \sqrt{2} & 0 \\ \frac{2}{\sqrt{2}} - \sqrt{2} & 0 & \sqrt{2} \end{bmatrix} = \left[\begin{array}{c|cc} 1 & \boxed{-1 \quad 0} \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] = Z$$

$$\Rightarrow \quad Z := [-1, 0]$$

$$\Rightarrow \quad H + G^* Z + Z^* G = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} [-1, 0]$$

$$\ldots + \begin{bmatrix} -1 \\ 0 \end{bmatrix} [0, -1] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1, 1]$$

$$\Rightarrow \quad C := [1, 1]$$

The necessary equation for optimality is thus

$$z = \begin{bmatrix} x \\ u \end{bmatrix}$$

$$0 = C\, z(t) = [1, 1] \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = x(t) + u(t)$$

$$\Rightarrow \quad u(t) = -x(t)$$

$\Rightarrow$ choose $\quad c_1 = c_2 = 1$

The closed-loop dynamics are

$\mathring{x}(t) = u(t) = x(t)$ which means

$$\underline{\underline{\mathring{x}(t) = x(t)}}\,.$$