# Equations of motion for an inverted double pendulum on a cart (in generalized coordinates) 

Consider a double pendulum which is mounted to a cart, as in the following graphic:


The length of the first rod is denoted by $l_{1}$ and the length of the second rod by $l_{2}$. The mass of the cart is denoted by $m$. We assume that the rods have no mass, that on the top of the first rod (and thus at the bottom of the second rod) there is a weight of mass $m_{1}$, and that on the top of the second rod there is a weight of mass $m_{2}$. All masses are assumed to be concentrated into a point.

We denote by $\theta_{1}=\theta_{1}(t)$ and $\theta_{2}=\theta_{2}(t)$ the deviation of the rods from the upright position at time $t \in \mathbb{R}$ as depicted in the image above. By $q=q(t)$ we denote the horizontal position of the cart and we assume that the cart cannot move vertically. The derivatives with respect to time are denoted by

$$
\frac{d}{d t} q(t)=\dot{q}, \quad \frac{d}{d t} \theta_{1}(t)=\dot{\theta}_{1}, \quad \frac{d}{d t} \theta_{2}(t)=\dot{\theta}_{2}
$$

The goal is to stabilize the pendulum in an upright position above the cart by only applying forces to the cart itself; think of only the cart having some kind of motor while the rods can dangle around freely. The control input $u=u(t)$ is thus the force that we can apply to the cart.

Furthermore, we assume that external distrubances $w_{1}, w_{2}, w_{3}$ act as forces on $q, \theta_{1}, \theta_{2}$; think of these external forces as wind or some human pushing the rods. The friction in the joints and the friction of the moving cart are modeled via a linear ansatz. We therefore introduce the damping coefficients $d_{1}, d_{2}, d_{3}$ and consider the friction/damping force of the cart to be $-d_{1} \dot{q}$ while the friction/damping forces in the joints are assumed to be $-d_{2} \dot{\theta}_{1}$ and $-d_{3} \dot{\theta}_{2}$.

The positions of the masses $m, m_{1}$, and $m_{2}$ are given by

$$
q_{0}:=\left[\begin{array}{l}
q \\
0
\end{array}\right], \quad q_{1}:=\left[\begin{array}{c}
q+l_{1} \sin \theta_{1} \\
l_{1} \cos \theta_{1}
\end{array}\right], \text { and } \quad q_{2}:=\left[\begin{array}{c}
q+l_{1} \sin \theta_{1}+l_{2} \sin \theta_{2} \\
l_{1} \cos \theta_{1}+l_{2} \cos \theta_{2}
\end{array}\right]
$$

respecitvely. Thus, the kinetic energy in the system is

$$
\begin{aligned}
K= & \frac{1}{2}\left\{m\left\|\dot{q}_{0}\right\|^{2}+m_{1}\left\|\dot{q}_{1}\right\|^{2}+m_{2}\left\|\dot{q}_{2}\right\|^{2}\right\} \\
= & \frac{1}{2}\left\{m \dot{q}^{2}+m_{1}\left[\left(\dot{q}+l_{1} \dot{\theta}_{1} \cos \theta_{1}\right)^{2}+\left(l_{1} \dot{\theta}_{1} \sin \theta_{1}\right)^{2}\right]+\right. \\
& \left.m_{2}\left[\left(\dot{q}+l_{1} \dot{\theta}_{1} \cos \theta_{1}+l_{2} \dot{\theta}_{2} \cos \theta_{2}\right)^{2}+\left(l_{1} \dot{\theta}_{1} \sin \theta_{1}+l_{2} \dot{\theta}_{2} \sin \theta_{2}\right)^{2}\right]\right\}
\end{aligned}
$$

and the potential energy can be given as

$$
P=g\left\{m_{1} l_{1} \cos \theta_{1}+m_{2}\left(l_{1} \cos \theta_{1}+l_{2} \cos \theta_{2}\right)\right\} .
$$

The principle of Lagrangian mechanics (as taught in "theoretical physics") states that to obtain the equations of motion for the cart, we have to define the Lagrangian $L:=K-P$ and then the equations of motion are

$$
\begin{aligned}
& u+w_{1}-d_{1} \dot{q}=\frac{d}{d t}\left\{\frac{\partial L}{\partial \dot{q}}\right\}-\left\{\frac{\partial L}{\partial q}\right\} \\
& =\frac{d}{d t}\left\{m \dot{q}+m_{1}\left(\dot{q}+l_{1} \dot{\theta}_{1} \cos \theta_{1}\right)+m_{2}\left(\dot{q}+l_{1} \dot{\theta}_{1} \cos \theta_{1}+l_{2} \dot{\theta}_{2} \cos \theta_{2}\right)\right\}-\{0\} \\
& =\quad\left(m+m_{1}+m_{2}\right) \ddot{q}+l_{1}\left(m_{1}+m_{2}\right) \ddot{\theta}_{1} \cos \theta_{1}-l_{1}\left(m_{1}+m_{2}\right)\left(\dot{\theta}_{1}\right)^{2} \sin \theta_{1} \\
& +m_{2} l_{2} \ddot{\theta}_{2} \cos \theta_{2}-m_{2} l_{2}\left(\dot{\theta}_{2}\right)^{2} \sin \theta_{2} \\
& =\quad\left(m+m_{1}+m_{2}\right) \ddot{q}+l_{1}\left(m_{1}+m_{2}\right) \ddot{\theta}_{1} \cos \theta_{1}+m_{2} l_{2} \ddot{\theta}_{2} \cos \theta_{2} \\
& -l_{1}\left(m_{1}+m_{2}\right)\left(\dot{\theta}_{1}\right)^{2} \sin \theta_{1}-m_{2} l_{2}\left(\dot{\theta}_{2}\right)^{2} \sin \theta_{2} \\
& w_{2}-d_{2} \dot{\theta}_{1}=\frac{d}{d t}\left\{\frac{\partial L}{\partial \dot{\theta}_{1}}\right\}-\left\{\frac{\partial L}{\partial \theta_{1}}\right\} \\
& =. \star=\left\{l_{1}\left(m_{1}+m_{2}\right) \dot{q} \dot{\theta}_{1} \sin \theta_{1}+l_{1} l_{2} m_{2} \dot{\theta}_{1} \dot{\theta}_{2} \sin \left(\theta_{1}-\theta_{2}\right)-g\left(m_{1}+m_{2}\right) l_{1} \sin \theta_{1}\right\} \\
& +\frac{d}{d t}\left\{l_{1}\left(m_{1}+m_{2}\right) \dot{q} \cos \theta_{1}+l_{1}^{2}\left(m_{1}+m_{2}\right) \dot{\theta}_{1}+l_{1} l_{2} m_{2} \dot{\theta}_{2} \cos \left(\theta_{1}-\theta_{2}\right)\right\} \\
& =\left\{l_{1}\left(m_{1}+m_{2}\right) \dot{q} \dot{\theta}_{1} \sin \theta_{1}+l_{1} l_{2} m_{2} \dot{\theta}_{1} \dot{\theta}_{2} \sin \left(\theta_{1}-\theta_{2}\right)-g\left(m_{1}+m_{2}\right) l_{1} \sin \theta_{1}\right\} \\
& +\left\{l_{1}\left(m_{1}+m_{2}\right) \ddot{q} \cos \theta_{1}+l_{1}^{2}\left(m_{1}+m_{2}\right) \ddot{\theta}_{1}+l_{1} l_{2} m_{2} \ddot{\theta}_{2} \cos \left(\theta_{1}-\theta_{2}\right)\right. \\
& \left.-l_{1}\left(m_{1}+m_{2}\right) \dot{q} \dot{\theta}_{1} \sin \theta_{1}-l_{1} l_{2} m_{2} \dot{\theta}_{2}\left(\dot{\theta}_{1}-\dot{\theta}_{2}\right) \sin \left(\theta_{1}-\theta_{2}\right)\right\} \\
& =\quad l_{1}\left(m_{1}+m_{2}\right) \ddot{q} \cos \theta_{1}+l_{1}^{2}\left(m_{1}+m_{2}\right) \ddot{\theta}_{1}+l_{1} l_{2} m_{2} \ddot{\theta}_{2} \cos \left(\theta_{1}-\theta_{2}\right) \\
& +l_{1} l_{2} m_{2}\left(\dot{\theta}_{2}\right)^{2} \sin \left(\theta_{1}-\theta_{2}\right)-g\left(m_{1}+m_{2}\right) l_{1} \sin \theta_{1} \\
& w_{3}-d_{3} \dot{\theta}_{2}=\frac{d}{d t}\left\{\frac{\partial L}{\partial \dot{\theta}_{2}}\right\}-\left\{\frac{\partial L}{\partial \theta_{2}}\right\} \\
& =. \star=\left\{-l_{2} m_{2} g \sin \theta_{2}+l_{2} m_{2} \dot{q} \dot{\theta}_{2} \sin \theta_{2}-l_{1} l_{2} m_{2} \dot{\theta}_{1} \dot{\theta}_{2} \sin \left(\theta_{1}-\theta_{2}\right)\right\} \\
& +\frac{d}{d t}\left\{l_{2}^{2} m_{2} \dot{\theta}_{2}+l_{2} m_{2} \dot{q} \cos \theta_{2}+l_{1} l_{2} m_{2} \dot{\theta}_{1} \cos \left(\theta_{1}-\theta_{2}\right)\right\} \\
& =\left\{-l_{2} m_{2} g \sin \theta_{2}+l_{2} m_{2} \dot{q} \dot{\theta}_{2} \sin \theta_{2}-l_{1} l_{2} m_{2} \dot{\theta}_{1} \dot{\theta}_{2} \sin \left(\theta_{1}-\theta_{2}\right)\right\} \\
& +\left\{l_{2}^{2} m_{2} \ddot{\theta}_{2}+l_{2} m_{2} \ddot{q} \cos \theta_{2}+l_{1} l_{2} m_{2} \ddot{\theta}_{1} \cos \left(\theta_{1}-\theta_{2}\right)\right. \\
& \left.-l_{2} m_{2} \dot{q} \dot{\theta}_{2} \sin \theta_{2}-l_{1} l_{2} m_{2} \dot{\theta}_{1}\left(\dot{\theta}_{1}-\dot{\theta}_{2}\right) \sin \left(\theta_{1}-\theta_{2}\right)\right\} \\
& =\quad l_{2} m_{2} \ddot{q} \cos \theta_{2}+l_{1} l_{2} m_{2} \ddot{\theta}_{1} \cos \left(\theta_{1}-\theta_{2}\right)+l_{2}^{2} m_{2} \ddot{\theta_{2}} \\
& -l_{1} l_{2} m_{2}\left(\dot{\theta}_{1}\right)^{2} \sin \left(\theta_{1}-\theta_{2}\right)-l_{2} m_{2} g \sin \theta_{2},
\end{aligned}
$$

where the MATLAB symbolic computations toolbox was used at the $=. \star$. $=$ symbols.
In matrix form and using the definition $y:=\left[\begin{array}{lll}q & \theta_{1} & \theta_{2}\end{array}\right]^{T}$ this yields

$$
\left.\begin{array}{rl} 
& \underbrace{\left[\begin{array}{ccc}
m+m_{1}+m_{2} & l_{1}\left(m_{1}+m_{2}\right) \cos \theta_{1} & m_{2} l_{2} \cos \theta_{2} \\
l_{1}\left(m_{1}+m_{2}\right) \cos \theta_{1} \\
l_{1}^{2}\left(m_{1}+m_{2}\right) & l_{1} l_{2} m_{2} \cos \left(\theta_{1}-\theta_{2}\right) \\
l_{2} m_{2} \cos \theta_{2}
\end{array}\right.}_{=: M(y)} \begin{array}{l}
l_{1} l_{2} m_{2} \cos \left(\theta_{1}-\theta_{2}\right)
\end{array} \\
l_{2}^{2} m_{2}
\end{array}\right] \underbrace{\left[\begin{array}{c}
\ddot{\ddot{\theta}} \\
\ddot{\theta}_{1} \\
\ddot{\theta}_{2}
\end{array}\right]}_{=: f(y, \dot{y}, u, w)} \underbrace{}_{\left.\begin{array}{c}
l_{1}\left(m_{1}+m_{2}\right)\left(\dot{\theta}_{1}\right)^{2} \sin \theta_{1}+m_{2} l_{2}\left(\dot{\theta}_{2}\right)^{2} \sin \theta_{2} \\
-l_{1} l_{2} m_{2}\left(\dot{\theta}_{2}\right)^{2} \sin \left(\theta_{1}-\theta_{2}\right)+g\left(m_{1}+m_{2}\right) l_{1} \sin \theta_{1} \\
l_{1} l_{2} m_{2}\left(\dot{\theta}_{1}\right)^{2} \sin \left(\theta_{1}-\theta_{2}\right)+g l_{2} m_{2} \sin \theta_{2}
\end{array}\right]-\left[\begin{array}{c}
d_{1} \dot{q} \\
d_{2} \dot{\theta}_{1} \\
d_{3} \dot{\theta}_{2}
\end{array}\right]+\left[\begin{array}{c}
u \\
0 \\
0
\end{array}\right]+\underbrace{\left[\begin{array}{c}
w_{1} \\
w_{2} \\
w_{3}
\end{array}\right]}_{=: w}}=
$$

or

$$
\begin{equation*}
M(y) \ddot{y}=f(y, \dot{y}, u, w) \tag{1}
\end{equation*}
$$

Since the determinate of $M(y)$ is

$$
\operatorname{det} M(y)=. \star=l_{1}^{2} l_{2}^{2} m_{2}(\underbrace{m m_{1}}_{>0}+\underbrace{m_{1}^{2} \sin ^{2} \theta_{1}+m_{1} m_{2} \sin ^{2} \theta_{1}+m m_{2} \sin ^{2}\left(\theta_{1}-\theta_{2}\right)}_{\geq 0})>0,
$$

for all $y \in \mathbb{R}^{3}$, we conclude that $M(y)$ is invertible. Thus we can rewrite (1) into the form $\ddot{y}=M^{-1}(y) f(y, \dot{y}, u, w)$ which with

$$
x:=\left[\begin{array}{l}
y \\
\dot{y}
\end{array}\right]
$$

and via order reduction gives

$$
\dot{x}=\frac{d}{d t}\left[\begin{array}{c}
y \\
\dot{y}
\end{array}\right]=\underbrace{\left[\begin{array}{c}
\dot{y} \\
M^{-1}(y) f(y, \dot{y}, u, w)
\end{array}\right]}_{=: F(x, u, w)}
$$

or in short notation the ODE (control) system

$$
\dot{x}=F(x, u, w) .
$$

