$\mathcal{L}e(P) \longrightarrow \mathcal{L}e(\lambda F + G) \longrightarrow$ (LTVS) $\dot{x}(t) = A(t)x(t) + B(t)u(t)$, $x(t_0) = x_0$

$\overset{FMM}{\Longrightarrow}$ $x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t}$

<u>Thm. 2.2:</u> Equivalent are:

i) (LTVS) is controllable from t_0 to t_1

ii) Every $z \in \mathcal{C}_\infty^n$ with $\begin{cases} \dot{z}(t) = A^*(t)z(t) \\ 0 = B^*(t)z(t) \end{cases}$

fulfills $z(t) = 0$ in $[t_0, t_1]$

iii) The Gramian of controllability is positive definite:

$$V(t_1, t_0) = \int_{t_0}^{t_1} \Phi(t_0, s)B(s)B^*(s)\Phi^*(t_0, s)\,ds > 0.$$

<u>Proof:</u> <u>i) $\Rightarrow$ ii):</u> last lecture

<u>ii) $\Rightarrow$ iii):</u> Since $V(t_1, t_0) = V^*(t_1, t_0) \geq 0$ (Homework S.4.T.1.a)) anyway it is sufficient to show that $V(t_0, t_1)$ is invertible. Therefor let $z_0 \in \operatorname{Kernel} V(t_0, t_1)$.

$\overset{S.4.T.1.b)}{\Longrightarrow}$ $B^*(t)\Phi^*(t_0, t)z_0 = 0 \quad \forall\, t \in [t_0, t_1]$.

Setting $\hat{z}(t) := \Phi^*(t_0, t)z_0$ we thus have $B^*(t)\hat{z}(t) = 0$ $\forall\, t \in [t_0, t_1]$ and using Lemma 1.25.c)

$\dot{\hat{z}}(t) = -A^*(t)\hat{z}(t)$, $\hat{z}(t_0) = z_0$.

By assumption this implies that $\hat{z}$ vanishes identically in $[t_0, t_1] \Rightarrow z_0 = \hat{z}_0(t_0) = 0$.

iii) $\Rightarrow$ i): Let $x_0, x_1 \in \mathbb{C}^n$ be arbitrary. Chose

$$u(t) := B^*(t)\,\Phi^*(t_0,t)\,V^{-1}(t_1,t_0)\left[\Phi(t_0,t_1)x_1 - x_0\right]$$

Then with Theorem 1.24. we see that the associated solution $x \in \mathcal{C}^n_\infty$ of (LTVS) with $x(t_0)=x_0$ satisfies

$$x(t_1) = \Phi(t_1,t_0)x_0 + \int_{t_0}^{t_1} \underbrace{\Phi(t_1,s)}_{\xrightarrow{\text{Lemma 1.25. a)}} = \Phi(t_1,t_0)\Phi(t_0,s)} B(s)\,u(s)\,ds$$

$$= \Phi(t_1,t_0)x_0 + \Phi(t_1,t_0)\overbrace{\left[\int_{t_0}^{t_1} \Phi(t_0,s)B(s)B^*(s)\Phi^*(t_0,s)\,ds\right]}^{=V(t_1,t_0)} \cdots$$

$$\cdot V^{-1}(t_1,t_0)\left[\Phi(t_0,t_1)x_1 - x_0\right]$$

$$= \cancel{\Phi(t_1,t_0)x_0} + \underbrace{\Phi(t_1,t_0)\Phi(t_0,t_1)}_{\xrightarrow{\text{Lemma 1.25 a)}} = I}x_1 - \cancel{\Phi(t_1,t_0)x_0} = x_1$$

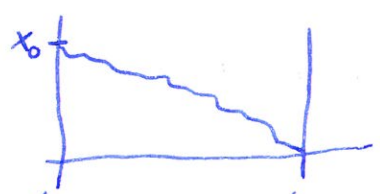
▨

The controllable and reachable subspace

Definition 2.3.

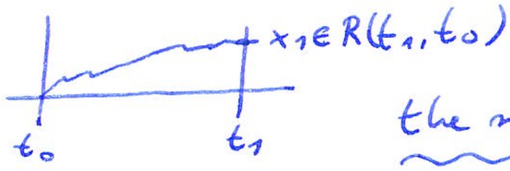
Let $t_0 < t_1$ and $A \in \mathcal{C}^{n,n}_\infty$, $B \in \mathcal{C}^{n,m}_\infty$. Then we call

$$C(t_1,t_0) := \left\{ x_0 \in \mathbb{C}^n \mid \exists (x,u) \in \mathcal{C}^{n+m}_\infty \text{ which solves } \dot{x}(t) = A(t)x(t) + B(t)u(t) \text{ and fulfills } \begin{array}{l} x(t_0)=x_0 \\ x(t_1)=0 \end{array} \right\}$$



the controllable subspace from t_0 to t_1 and

$$R(t_1,t_0) := \left\{ x_1 \in \mathbb{C}^n \mid \exists (x,u) \in \mathcal{C}^{n+m}_\infty \text{ which solves } \dot{x}(t) = A(t)x(t) + B(t)u(t) \text{ and fulfills } x(t_0)=0,\ x(t_1)=x_1 \right\}$$



the reachable subspace from t_0 to t_1

Theorem 2.4. Let $A \in \mathcal{C}_\infty^{n,n}$ and $B \in \mathcal{C}_\infty^{n,m}$, $t_0 < t_1$. Let Φ be the fundamental matrix of $\dot{x} = A(t)x$ and let $V(t_0,t_1)$ be the Gramian of controllability. Then we have:

i) $C(t_1,t_0) = \qquad \text{image } V(t_0,t_1)$

ii) $R(t_1,t_0) = \Phi(t_1,t_0) \cdot \text{image } V(t_0,t_1)$

Proof: i) Similar to ii) $\Rightarrow$ Homework

ii) $[x_1 \in R(t_0,t_1)] \quad \Longleftrightarrow$

$[\exists (x,u) \in \mathcal{C}_\infty^{n+m}$ with $\dot{x} = A(t)x(t) + B(t)u(t)$ and $x(t_0) = 0 \ldots$ and $x(t_1) = x_1]$

Thm. 1.24

$\Longleftrightarrow [\exists (u) \in \mathcal{C}_\infty^{m}$ with $x_1 = x(t_1) = \Phi(t_1,t_0)\underbrace{0}_{=x_0=x(t_0)} + \int_{t_0}^{t_1} \Phi(t_1,s)B(s)u(s)\,ds$

$= \Phi(t_1,t_0) \cdot \int_{t_0}^{t_1} \Phi(t_0,s)B(s)u(s)\,ds\ ]$

$\Longleftrightarrow [x_1 \in \Phi(t_1,t_0) \cdot \{\hat{x} \in \mathbb{C}^n \mid \exists (u) \in \mathcal{C}_\infty^m \text{ with } \hat{x} = \int_{t_0}^{t_1} \Phi(t_0,s)B(s)u(s)\,ds\}$

$\overset{S.4.T.1.c)}{=} \Phi(t_1,t_0) \cdot \text{image } V(t_1,t_0).\ ]$ ▨

The controllability matrix

For linear time-invariant state-space systems (LTIS) $\dot{x}(t) = Ax(t) + Bu(t)$ with $A \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$, $x \in \mathcal{C}_\infty^n$, $u \in \mathcal{C}_\infty^m$ the fundamental matrix only depends on $t_1 - t_0$, i.e., we have $\Phi(t_1,t_0) = e^{(t_1-t_0)\cdot A} = e^{((t_1-t_0)-0)A} = \Phi(t_1-t_0,0)$ and thus also the Gramian of controllability only depends on $t_1 - t_0$: $V(t_1,t_0) = \int_{t_0}^{t_1} \Phi(t_0,s)BB^*\Phi^*(t_0,s)\,ds$

$= \int_{t_0}^{t_1} \dot{\varphi}(s)\Phi(t_0,\underbrace{s-t_0+t_0}_{=:\varphi(s)=r})BB^*\Phi(t_0,s-t_0,t_0)\,ds$

$$\overset{\text{Substitution}}{=} \int_{\psi(t_0)}^{\psi(t_1)} \Phi(t_0, r+t_0)\, BB^* \Phi(t_0, r+t_0)\, dr$$

$$= \int_0^{t_1-t_0} \Phi(0, r+t_0-t_0)\, BB^* \Phi^*(0, r+t_0-t_0)\, dr$$

$$= V(\underbrace{t_1-t_0}_{=:\tau}, 0) = \int_0^{\tau} e^{-sA} BB^* e^{-sA^*}\, ds.$$

The same is true for the controllable and reachable sets $C(t_1,t_0) = C(t_1-t_0, 0)$, $R(t_1,t_0) = R(t_1-t_0, 0)$.

Thus for time-invariant systems (LTIS) we define

$$V(\tau) := V(\tau, 0)$$
$$C(\tau) := C(\tau, 0)$$
$$R(\tau) := R(\tau. 0)$$

Theorem 2.4. then implies $C(\tau) = \text{image } V(\tau)$
$R(\tau) = e^{A\cdot\tau} \cdot \text{image } V(\tau)$.

Definition 2.5.

Let $A \in \mathbb{C}^{n,n}$ and $B \in \mathbb{C}^{n,m}$. Then the matrix

$K(A,B) := [B, AB, \ldots, A^{n-1}B] \in \mathbb{C}^{n,n\cdot m}$ is called

the (Kalman) controllability matrix (associated with A and B)

Theorem 2.6.

Let $A \in \mathbb{C}^{n,n}$ and $B \in \mathbb{C}^{n,m}$, and $\tau > 0$. Then we have

image $K(A,B) =$ image $V(\tau)$.

Furthermore, if all eigenvalues of A have positive real part, then the improper Integral $\int_0^\infty e^{-sA} BB^* e^{-sA^*}\, ds =: V(\infty)$ is well defined.

In this case we also have

$$\text{image } K(A,B) = \text{image } V(\infty).$$

<u>Proof:</u> We show that the orthogonal complements are equal:

$$(\text{image } K(A,B))^{\perp} = (\text{image } V(\tau))^{\perp} \quad \forall \tau \in (0,\infty]$$

We have

$$(1) \begin{cases} z_0 \in (\text{image } V(\tau))^{\perp} \Leftrightarrow \\ z_0^* V(\tau)\alpha = \langle z_0, V(\tau)\alpha \rangle = 0 \quad \forall \alpha \in \mathbb{C}^m \quad \overset{V(\tau)=V(\tau)^*}{\Leftrightarrow} \\ V^*(\tau) z_0 = 0 \quad \Leftrightarrow \text{ S.4. T. 1. b)} \\ B^* e^{-sA^*} z_0 = 0 \quad \text{in } [0,\tau] \quad \Leftrightarrow \\ z_0^* e^{-sA} B = 0 \quad \text{in } [0,\tau] \end{cases}$$

Differencing the last equation k-times shows

$$0 = \left(\frac{\partial}{\partial t}\right)^k (z_0^* e^{-sA} B) = z_0^* (-A)^k e^{-sA} B, \quad \forall s \in [0,\tau]$$

Choosing $s=0$ implies $z_0^* A^k B = 0$, $\forall k \in \mathbb{N}_0$.

This in turn implies that

$$0 = \sum_{i=0}^{\infty} z_0^* A^i B \frac{(-s)^i}{i!} = z_0^* \left(\sum_{i=0}^{\infty} \frac{(-sA)^i}{i!}\right) \cdot B = z_0^* e^{-sA} \cdot B \quad \forall s \in [0,\tau]$$

which again is the last equation in (1). Thus, we have that $[z_0^* e^{-sA} B = 0 \text{ in } [0,\tau]]$

$\Leftrightarrow [z_0^* A^k B = 0 \quad \forall k \in \mathbb{N}]$ (2).

By Cayley-Hamilton we know that there exist $\alpha_0, \dots, \alpha_{m-1}$ with $A^m = \alpha_{m-1} A^{m-1} + \dots + \alpha_1 A^1 + \alpha_0 \overset{I}{A^0}$ which means

$$[z_0^* A^k B = 0 \quad \forall k \in \{0,\dots,m-1\}] \Leftrightarrow [z_0 A^k B = 0 \quad \forall k \in \mathbb{N}_0] \quad (3)$$

We conclude that

$$z_0 \in (\operatorname{image} V(\tau))^{\perp} \overset{(1)}{\Longleftrightarrow}$$

$$z_0^* e^{-sA} B = 0 \text{ in } [0, \tau] \overset{(2)}{\Longleftrightarrow}$$

$$z_0^* A^k B = 0 \ \forall k \in \mathbb{N}_0 \overset{(3)}{\Longleftrightarrow}$$

$$z_0^* A^k B = 0 \ \forall k \in \{0, \dots, n-1\} \Longleftrightarrow$$

$$z_0^* [B, AB, \dots, A^{n-1} B] = 0 \Longleftrightarrow$$

$$z_0^* K(A, B) = 0 \Longleftrightarrow$$

$$z_0 \perp \operatorname{image} K(A, B).$$

Corollary 2.7:

For linear time-invariant state-space systems $\dot{x}(t) = Ax(t) + Bu(t)$ with $A \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$ the controllable and reachable ~~system~~ subspaces do not depend on $\tau \in (0,\infty)$, i.e., we have

$$C(\tau_1) = C(\tau_2) \text{ and } R(\tau_1) = R(\tau_2)$$

for all $\tau_1, \tau_2 > 0$

Proof: We have

$$C(\tau_1) \overset{\text{Thm 2.4 i)}}{=} \text{image } V(\tau_1) \overset{\text{Thm 2.6}}{=} \text{image } K(A,B)$$
$$= \text{image } V(\tau_2) = C(\tau_2)$$

and similar with the substitution $\varphi(s) = \tau_1 - s \; (=: r)$, $\dot{\varphi}(s) = -1$ we have

$$R(\tau_1) \underset{\text{Thm 2.4 ii)}}{=} e^{A\tau_1} \cdot \text{image } V(\tau_1)$$

$$= \text{image } \Big(e^{A\tau_1} \underbrace{\int_0^{\tau_1} e^{-sA} BB^* e^{-sA^*} \, ds}_{= V(\tau_1)} \underbrace{e^{A^*\tau_1}}_{\substack{\text{is inv.} \\ \text{Lemma 1.25, 2)}}} \Big)$$

$$= \text{image } (-1) \int_0^{\tau_1} \dot{\varphi}(s) \, e^{(\tau_1 - s)A} BB^* e^{\overbrace{(\tau_1 - s)}^{\varphi(s)} A^*} \, ds$$

$$\overset{\text{substitution}}{=} \text{image } (-1) \int_{\varphi(0) = \tau_1}^{\varphi(\tau_1) = 0} e^{rA} BB^* e^{rA^*} \, dr$$

$$= \text{image } \int_0^{\tau_1} e^{rA} BB^* e^{rA^*} \, dr$$

$$\overset{\text{Thm 2.6}}{=} \text{image } K(A,B) \overset{\text{as above}}{=} \ldots = R(\tau_2)$$, which proves the claim ☑

If $K(A,B)$ has full row rank then we have

$$C(\tau) \underset{\text{Thm 2.4 i)}}{=} \text{image } V(\tau) \underset{\text{Thm 2.6}}{=} \text{image } K(A,B) = \mathbb{C}^n$$

and also

$$R(\tau) \underset{\text{Thm 2.4. ii)}}{=} e^{A\tau} \cdot \text{image } V(\tau) \underset{\text{Thm 2.6.}}{=} \underset{\substack{\text{invertible} \\ \text{R.1.25, 2)}}}{e^{A\tau}} \cdot \text{image } K(A,B)$$

$$= \left(e^{A\tau_1}\right)\mathbb{C}^n = \mathbb{C}^n.$$

Definition 2.8 The matrix pair $(A,B) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,m}$ is called *controllable* if $\operatorname{rank} K(A,B) = n$.

The Kalman decomposition

Definition 2.9

Let $A \in \mathbb{C}^{n,n}$ and $\mathcal{R} \subseteq \mathbb{C}^n$ be a linear subspace.
Then we say that $\mathcal{R}$ is *A-invariant* if
$[x_0 \in \mathcal{R}] \Rightarrow [Ax_0 \in \mathcal{R}]$.
In short notation: $A\mathcal{R} \subseteq \mathcal{R}$.

Lemma 2.10

Let $(A,B) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,m}$. Then

$$\mathcal{R} := \text{image}(K(A,B))$$

is the smallest A-invariant subspace which contains (image B), i.e. we have

i) $A\mathcal{R} \subseteq \mathcal{R}$

ii) Every A-invariant subspace $\mathcal{V} \subseteq \mathbb{C}^n$ with image $B \subseteq \mathcal{V}$ satisfies $\mathcal{R} \subseteq \mathcal{V}$.

Proof:

i) Let $x_0 \in \text{image } K(A,B)$, i.e, there exist $\beta_0, \dots, \beta_{n-1} \in \mathbb{C}^m$ with $x_0 \in B\beta_0 + AB\beta_1 + \dots + A^{n-1}B\beta_{n-1}$

$\Rightarrow Ax_0 = AB\beta_0 + A^2B\beta_1 + \dots + A^nB\beta_{n-1}$

Since by Cayley-Hamilton A^nB can be written as a linear combination of $I \cdot B, AB, \dots, A^{n-1}B$ (compare proof of theorem 2.6) we have that

$A^nB\beta_{n-1} \in \text{image } K(A,B)$ and thus also

$Ax_0 \in \text{image } K(A,B)$.

ii) Since $(\text{image } B) \subseteq \mathcal{W}$ and $\mathcal{W}$ is A-invariant we have

$\text{image } (AB) = A \cdot \text{image } (B) \subseteq \mathcal{W}$

$\Rightarrow \text{image } (A^2B) = A \cdot \text{image } (AB) \subseteq \mathcal{W}$

$\Rightarrow \dots \Rightarrow \text{image } (A^{n-1}B) = A \cdot \text{image}(A^{n-2}B) \subseteq \mathcal{W}$

$\Rightarrow \text{image } K(A,B) = (\text{image } B) + (\text{image } AB) + \dots + (\text{image } A^{n-1}B) \subseteq \mathcal{W}$

$\boxtimes$

Theorem 2.11: (Kalman decomposition of controllability)

Let $(A,B) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,m}$ and set $r := \text{rank } K(A,B)$.

Then there exists a unitary matrix $V \in \mathbb{C}^{n,n}$ such that

$$V^*AV = \begin{array}{c} \left[\begin{array}{cc} A_1 & A_2 \\ 0 & A_3 \end{array}\right] \begin{array}{l} \}r \\ \}n-r \end{array} \\ \begin{array}{cc} \underbrace{\hphantom{A_1}}_{r} & \underbrace{\hphantom{A_3}}_{n-r} \end{array} \end{array}$$

and $V^*B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{array}{l} \}r \\ \}n-r \end{array}$ where (A_1, B_1) is controllable.

Proof: Let $v_1, \dots, v_r$ and $v_{r+1}, \dots, v_n$ be orthogonal bases ~~bases~~ of $\mathcal{R} := \text{image}(A,B)$ and $\mathcal{R}^\perp$, respectively.

Set $V_1 = [v_1, \dots, v_r]$ and $V_2 = [v_{r+1}, \dots, v_n]$.

Then $V := [V_1, V_2]$ is unitary and we have $V_2^* K(A,B) = 0$.

Since $K(A,B) = [B, AB, \dots]$ we have

$$V^*B = \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} B =: \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \begin{array}{l} \}r \\ \}n-r \end{array}.$$

Since $\mathcal{R}$ is A-invariant (Lemma 2.9) we have
image $(AV_1) = A \cdot$ image $V_1 = A\mathcal{R} \subseteq \mathcal{R}$
which means that there exists a matrix $\Lambda \in \mathbb{C}^{r,r}$
with $AV_1 = V_1 \Lambda$

$$\Rightarrow V_2^* A V_1 = \underbrace{V_2^* V_1}_{=0} \Lambda = 0$$

Thus $V^* A V = \begin{bmatrix} V_1^* \\ V_2^* \end{bmatrix} A [V_1, V_2] =: \begin{bmatrix} V_1^* A V_1 & V_1^* A V_2 \\ V_2^* A V_1 & V_2^* A V_2 \end{bmatrix}$

$$=: \begin{bmatrix} A_1 & A_2 \\ 0 & A_3 \end{bmatrix}.$$

Furthermore the equation

$$V^* A^k B = (V^* A V)^k V^* B = \begin{bmatrix} A_1^k & * \\ 0 & A_3^k \end{bmatrix} \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} A_1^k B_1 \\ 0 \end{bmatrix}, \quad k = 0, 1, \dots, n-1$$

implies $\quad V^* K(A,B) = [V^* B, V^* AB, \dots, V^* A^{n-1} B]$

$$= \begin{bmatrix} K(A_1, B_1) & , A_1^r B_1 & , \dots, & A_1^{n-1} B_1 \\ 0 & , \quad 0 & , \dots, & 0 \end{bmatrix}.$$

Since by Cayley-Hamilton the matrix A_1^r
(and thus also A^k with $k > r$) is a linear
combination of $A_1^0, A_1^1, \dots, A_1^{r-1}$ we conclude that
rank $K(A_1, B_1) =$ rank $(V^* K(A,B)) =$ rank $(K(A,B)) = r$,
and thus the claim.

$\blacksquare$

Controllability of behaviors

Definition 2.12.

Let $\mathfrak{B} \subseteq \mathcal{W} := \{z : \mathbb{R} \to \overset{\text{some vector space}}{W}\}$ be a continuous-time dynamic behavior. Then we say that $\mathfrak{B}$ is controllable from t_0 to t_1 if for all $z_0, z_1 \in \mathfrak{B}$ there exists a $z \in \mathfrak{B}$ with

$$z(t) = \begin{cases} z_0(t) & , t \leq t_0 \\ z_1(t) & , t \geq t_1 \end{cases}$$

Theorem 2.13.

Let $P \in \mathbb{C}[\lambda]^{p,q}$ and $t_0 < t_1$. Then the following are equivalent:

1) $\mathfrak{B}(P)$ is controllable from t_0 to t_1
2) $\mathfrak{Z}(P) = \emptyset$ ($\Leftrightarrow \operatorname{rank}_{\mathbb{C}(\lambda)} P = \operatorname{rank} P(\lambda_0)$, $\forall \lambda_0 \in \mathbb{C}$ L.1.9.)
3) $\mathfrak{B}(P)$ admits an image representation, i.e., there exists an $U \in \mathbb{C}[\lambda]^{q,m}$ such that $\mathfrak{B}(P) = \operatorname{image}_{\mathcal{C}_\infty} U(\frac{d}{dt}) := \{z \in \mathcal{C}_\infty^q \mid \exists\, \alpha \in \mathcal{C}_\infty^m$ such that $z = U(\frac{d}{dt}) \cdot \alpha\}$
4) Every right prime polynomial kernel spanning matrix U induces an image representation of $\mathfrak{B}(P)$.
5) The Kronecker canonical from the canonical linearization does not contain any blocks of type $\mathcal{J}$ (cf. handout "First order systems")