Theorem 2.21

Let $A \in \mathcal{C}_\infty^{n,n}$, $C \in \mathcal{C}_\infty^{p,n}$, $t_0 < t_1$, and Φ be the fundamental Matrix of $\dot{x} = A(t)x$. Let $\mathcal{L}$ be defined by (*).

With $B \in \mathcal{C}_\infty^{n,m}$, $D \in \mathcal{C}_\infty^{p,m}$ define the behavior with an additional input

$$\tilde{\mathcal{L}} := \left\{ (y,x,u) \in \mathcal{C}_\infty^{p+n+m} \;\middle|\; \begin{matrix} \dot{x} = A(t)x + B(t)u \\ y = C(t)x + D(t)u \end{matrix} \right\}.$$

Then the following are equivalent:

1) $\mathcal{L}$ is observable in $[t_0, t_1]$.

2) For all trajectories $(y,x) \in \mathcal{L}$ we have
$$\left[y(t) = 0 \;\; \forall t \in [t_0,t_1] \right] \Rightarrow \left[x(t) = 0 \;\; \forall t \in [t_0,t_1] \right]$$

3) For all trajectories $(y_1, x_1, u_1), (y_2, x_2, u_2) \in \tilde{\mathcal{L}}$ we have $\left[y_1(t) = y_2(t),\, u_1(t) = u_2(t) \;\; \forall t \in [t_0, t_1] \right]$
$\Rightarrow \left[x_1(t) = x_2(t) \;\; \forall t \in [t_0, t_1] \right]$

4) The system $\dot{x}(t) = -A^*(t)\, x(t) + C^*(t)\, v(t)$ is controllable

5) The Gramian of observability is positive definite:
$$W(t_1, t_0) := \int_{t_0}^{t_1} \Phi^*(s,t_0)\, C^*(s)\, C(s)\, \Phi(s,t_0)\, ds > 0.$$

Proof: 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3) follows from the linearity of the system.

2.) $\Leftrightarrow$ 4.) $\Leftrightarrow$ 5) follows from Theorem 2.2.

Homework! $\boxtimes$

The unobservable subspace

Definition 2.22: Let $t_0 < t_1$ and $A \in \mathcal{C}_\infty^{n,n}$, $C \in \mathcal{C}_\infty^{p,n}$, then we call $\mathcal{U}(t_1,t_0) := \{x_0 \in \mathbb{C}^n \mid$ the unique solution of $\dot{x} = A(t)x$, $x(t_0) = x_0$ satisfies $C(t)x(t) \equiv 0$ in $[t_0,t_1]\}$

The *unobservable subspace from t_0 to t_1*.

Theorem 2.23:
Let $A \in \mathcal{C}_\infty^{n,n}$, $C \in \mathcal{C}_\infty^{p,n}$, $t_0 < t_1$. Then
$\mathcal{U}(t_0,t_1) = \text{kernel } W(t_1,t_0)$ where $W(t_1,t_0)$ is the Gramian of observability from Theorem 2.21.

Proof: Let $\Phi(t,t_0)$ denote the fundamental matrix of $\dot{x} = A(t)x$. Then the unique solution of $\dot{x} = A(t)x$, $x(t_0) = x_0$ is $x(t) = \Phi(t,t_0)x_0$ (by Thm. 1.24) and thus we have

$$[x_0 \in \mathcal{U}(t_1,t_0)] \Leftrightarrow [C(t)\Phi(t,t_0)x_0 \equiv 0 \text{ in } [t_0,t_1]]$$

$$\overset{\text{Series 4 Task 1.b)}}{\Longleftrightarrow} [x_0 \in \text{kernel} \int_{t_0}^{t_1} \Phi^*(t,t_0)C^*(t)C(t)\Phi(t,t_0)\,dt \overset{\text{kernel}}{=} W(t_1,t_0)]$$

$\blacksquare$

The observability Matrix

In the time invariant case $(*)$ $\begin{cases} \dot{x}(t) = Ax(t) \\ y(t) = Cx(t) \end{cases}$,

with $A \in \mathbb{C}^{n,n}$, $C \in \mathbb{C}^{p,n}$, $x \in \mathcal{C}_\infty^n$, $y \in \mathcal{C}_\infty^p$, the Gramian of observability only depends on $t_1 - t_0$, i.e, we have:

$$W(t_1,t_0) = \ldots = W(t_1-t_0,0) = \int_0^{\tau} e^{sA^*} C^* C e^{sA}\, ds.$$

The same is true for the unobservable set

$$\mathcal{U}(t_1,t_0) = \mathcal{U}(t_1-t_0,0).$$

Thus for time-invariant systems $(*)$ we define

$$W(\tau) := W(\tau,0), \quad \mathcal{U}(\tau) := \mathcal{U}(\tau,0).$$

Theorem 2.23 then implies $\mathcal{U}(\tau) = \text{kernel } W(\tau)$.

<u>Definition 2.24:</u> Let $A \in \mathbb{C}^{n,n}$, $C \in \mathbb{C}^{p,n}$ then the matrix

$$K_o(A,C) := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \in \mathbb{C}^{n\cdot p,n}$$ is called the (Kalman) observability matrix

<u>Theorem 2.25:</u> Let $A \in \mathbb{C}^{n,n}$, $C \in \mathbb{C}^{p,n}$, and $\tau > 0$.
Then we have $\text{kernel } K_o(A,C) = \text{kernel } W(\tau)$.

Furthermore, if all eigenvalues of A have negative real part, then the improper integral

$$\int_0^{\infty} e^{sA^*} C^* C e^{sA}\, ds =: W(\infty)$$ is well defined.

In this case we also have

$$\text{Kernel } K_o(A,C) = \text{Kernel } W(\infty)$$

<u>Proof:</u> Since for any matrix $M \in \mathbb{C}^{p,q}$ we have

$$(\text{kernel } M^*) = (\cancel{\text{kernel}}\ \text{image } M)^{\perp} \quad ⓐ$$

and we also have $$K_o(A,C) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = [C^*, A^*C^*, \ldots]^*$$

$$= (K_c(A^*, C^*))^* \quad \text{where } K_c(A,B) := K(A,B)$$

denotes the Kalman controllability matrix, we conclude that

$$
\begin{aligned}
&\text{kernel } K_o(A,C) = \text{kernel } (K_o^*(A,C))^* \\
&\overset{ⓐ}{=} (\text{image } K_o^*(A,C))^\perp = (\text{image } K_c(A^*,C^*))^\perp \\
&\overset{\substack{\text{series 4}\\ \text{Task 5}}}{=} (\text{image } K_c(-A^*,C^*))^\perp \underset{\text{Thm. 2.6.}}{=} \left(\text{image} \int_0^\tau e^{-s(-A^*)} C^* C e^{-s(-A^*)} ds\right)^\perp \\
&= \left(\text{image} \int_0^\tau e^{sA^*} C^* C e^{sA} ds\right)^\perp = (\text{image } W(\tau))^\perp \\
&\overset{ⓐ}{=} \text{kernel } W(\tau)^* \underset{W(\tau)=W^*(\tau)}{=} \text{kernel } W(\tau),
\end{aligned}
$$

which proves the claim ∎

Theorem 2.25 especially implies that the uncontrollable subspace does not depend on τ, i.e., for all $\tau_1, \tau_2 > 0$ we have $U(\tau_1) = U(\tau_2)$.

If $K_o(A,C)$ has full column rank then we have

$$U(\tau) = \text{kernel } W(\tau) = \text{kernel } K_o(A,C) = \{0\}.$$

<u>Definition 2.26:</u> The matrix pair $(A,C) \in \mathbb{C}^{n,n} \times \mathbb{C}^{p,n}$ is called <u>observable</u> if $\text{rank } K_o(A,C) = n$

<u>The Kalman decomposition of observability</u>

<u>Theorem 2.27:</u> Let $(A,C) \in \mathbb{C}^{n,n} \times \mathbb{C}^{p,n}$ and set $r := \text{rank } K_o(A,C)$. Then there exists a unitary matrix $V \in \mathbb{C}^{n,n}$ such that

$$
V^* A V = \underbrace{\left[\begin{array}{c|c} A_1 & 0 \\ \hline A_2 & A_3 \end{array}\right]}_{r \quad\; n-r}
\begin{array}{l} \}\, r \\ \}\, n-r \end{array}, \quad CV = [\underbrace{C_1}_{r}, \underbrace{0}_{n-r}], \text{ where}
$$

(A_1, C_1) is observable.

Proof: Since $r = \operatorname{rank} K_c(A^*, C^*)$, let $V \in \mathbb{C}^{n,n}$ be unitary (with thm. 2.11) such that

$$V^* A^* V =: \begin{bmatrix} A_1^* & A_2^* \\ 0 & A_3^* \end{bmatrix}, \quad V^* C^* =: \begin{bmatrix} C_1 \\ 0 \end{bmatrix} \quad \text{such that } (A_1^*, C_1^*)$$

is controllable. Then (A_1, C_1) is observable and $V^* A V$, CV have the desired form. $\boxtimes$

Observability and the Lyapunov equation

Corollary 2.28: Let $A \in \mathbb{C}^{n,n}$ be with $\sigma(A) \subseteq \mathbb{C}_-$ and let $C \in \mathbb{C}^{p,n}$. Then (A, C) is observable if and only if the unique Hermitian solution $Y = Y^* \in \mathbb{C}^{n,n}$ of the Lyapunov equation

$$A^* Y + Y A = -C^* C$$

is positive definite: $Y > 0$.

Proof: Analogous to Corollary 2.17 $\blacksquare$

Observability of behaviors

Definition 2.29:

Let $R \in \mathbb{C}[\lambda]^{p,q}$, $M \in \mathbb{C}[\lambda]^{p,r}$. For the latent variable description $\mathcal{L}_f := \{(z, \ell) \in \mathcal{C}_\infty^{q+r} \mid R(\frac{\partial}{\partial t}) z = M(\frac{\partial}{\partial t}) \ell\}$ we say that ℓ is observable from z, if for all $(z, \ell_1), (z, \ell_2) \in \mathcal{L}_f$ we have $\ell_1 = \ell_2$.

Theorem 2.30: With the notation from Definition 2.29. we have that the following are equivalent:

1.) ℓ is observable from z.

2.) $\mathfrak{L}_e(M) = \{0\}$.

3.) M is right prime.

Proof: 1.) $\Rightarrow$ 2.) Let $\ell \in \mathfrak{L}_e(M)$. Then $(0,\ell) \in \mathfrak{L}_{e_f}$. ~~Since also $(0,0) \in \mathfrak{L}_{e_f}$~~. Since also $(0,0) \in \mathfrak{L}_{e_f}$ the assumption implies $0 = \ell$. This shows $\mathfrak{L}_e(M) \subseteq \{0\}$. The other inclusion "$\supseteq$" is also true.

2.) $\Rightarrow$ 1.) Let $(z,\ell_1),(z,\ell_2) \in \mathfrak{L}_{e_f}$. This means that

$$R(\tfrac{\partial}{\partial t})z = M(\tfrac{\partial}{\partial t})\ell_1, \quad R(\tfrac{\partial}{\partial t})z = M(\tfrac{\partial}{\partial t})\ell_2$$

$$\Rightarrow M(\tfrac{\partial}{\partial t})\ell_1 = M(\tfrac{\partial}{\partial t})\ell_2 \Rightarrow M(\tfrac{\partial}{\partial t})(\ell_2-\ell_1) = 0 \Rightarrow \ell_2-\ell_1 = 0$$

$\Rightarrow \ell_1 = \ell_2$. Thus ℓ is observable from z.

2.) $\Leftrightarrow$ 3.) is Homework Series 5. Task 5.

$\blacksquare$

Corollary 2.31: (Hautus-Test for observability)
Let $A \in \mathbb{C}^{n,n}, B \in \mathbb{C}^{n,m}, C \in \mathbb{C}^{p,n}, D \in \mathbb{C}^{p,m}$ and consider the latent variable description

$$\mathfrak{L}_{e_f} := \left\{ (x,y,u) \in \mathcal{C}_\infty^{n+p+m} \;\middle|\; \begin{bmatrix} \frac{\partial}{\partial t}I - A \\ -C \end{bmatrix} \overset{\text{latent}}{x} = \begin{bmatrix} B & 0 \\ D & -I \end{bmatrix} \overset{\text{manifest}}{\begin{bmatrix} u \\ y \end{bmatrix}} \right\}$$

Then (A,C) is observable if and only if in $\mathfrak{L}_{e_f}$ the latent variable x can be observed from $\underbrace{(y,u)}_{\text{the manifest variable}}$. This is equivalent to $\begin{bmatrix} \lambda I - A \\ -C \end{bmatrix}$ being right prime (by Theorem 2.30).

Proof: [x is observable from (y,u)]

$\overset{\text{Thm. 2.30}}{\Longleftrightarrow}$ $\left[\begin{bmatrix} \lambda I - A \\ -C \end{bmatrix} \text{ is right prime} \right]$

$\overset{\text{series 5. Task 5.}}{\Longleftrightarrow}$ $\left[\mathcal{L}_e\left(\begin{bmatrix} \lambda I - A \\ -C \end{bmatrix} \right) = \{0\} \right]$

$\Longleftrightarrow$ $\left[\forall x \in \mathcal{L}_e(\lambda I - A) \text{ for which } y := -Cx = 0 \text{ we have } x = 0 \right]$

$\overset{\text{Thm. 2.21 1.) 2.) \& 5.)}}{\Longleftrightarrow}$ $(\square)$ $\left[\operatorname{rank} W(t_1, t_0) = n \right]$ $\overset{\text{Thm 2.25}}{\Longleftrightarrow}$ $\left[\operatorname{rank} K_o(A, C) = n \right]$

$\overset{\text{Def. 2.26}}{\Longleftrightarrow}$ $\left[(A, C) \text{ is observable} \right]$.

It would also be possible to use the other equivalent characterizations of Theorem 2.21 at $(\square)$ which, of course, would change the proof. ∎