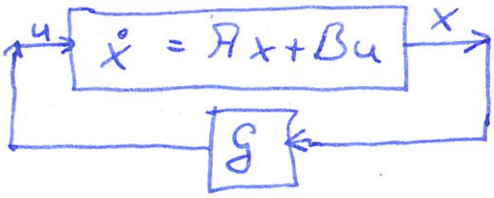
Stabilization via State-feedback

Let $A \in \mathbb{C}^{n,n}$, $B \in \mathbb{C}^{n,m}$ and consider the system (LTIS)

$$\dot{x}(t) = Ax(t) + Bu(t).$$

If it would be possible to measure the whole state $x(t)$, one could try to choose the input $u(t)$ depending on $x(t)$. In particular, making a linear ansatz, one could try to determine a so called ~~state-feedback matrix/gain~~ $G \in \mathbb{C}^{m,n}$ and choose $u(t) = Gx(t)$



Thus for the closed loop system we have

$$\dot{x}(t) = Ax(t) + Bu(t) = Ax(t) + BGx(t) = (A+BG)x(t)$$

which corresponds to the behavior $\mathfrak{L}(\lambda I - (A+BG))$.

From another perspective, we consider the closed-loop behavior

$$\mathfrak{L} := \left\{ (x,u) \in \mathcal{C}_\infty^{n+m} \;\middle|\; \begin{array}{l} \dot{x} = Ax + Bu \\ u = Gx \end{array} \right\} = \left\{ (x,u) \in \mathcal{C}_\infty^{n+m} \;\middle|\; \begin{array}{l} (\frac{d}{dt} I - A)x - Bu = 0 \\ -Gx + Iu = 0 \end{array} \right\}$$

$$= \mathfrak{L}\left(\begin{bmatrix} \lambda I - A & -B \\ -G & I \end{bmatrix} \right) = \mathfrak{L}\left(\begin{bmatrix} I & B \\ & I \end{bmatrix} \begin{bmatrix} \lambda I - A & -B \\ -G & I \end{bmatrix} \right)$$

$$\overset{(\square)}{=} \mathfrak{L}\left(\begin{bmatrix} \lambda I - (A+BG) & 0 \\ -G & I \end{bmatrix} \right) \overset{\text{Homework, S8, T.3}}{=} \begin{bmatrix} I \\ G \end{bmatrix} \mathfrak{L}(\lambda I - (A+BG))$$

The problem of stabilization via state feedback is then to determine a $G \in \mathbb{C}^{m,n}$ such that

$$(*) \begin{cases} \mathcal{L}(\lambda I - (A+BG)) & \text{is stable} \overset{(\square)}{\iff} \\ \mathcal{L}\left(\begin{bmatrix} \lambda I - A & -B \\ -G & I \end{bmatrix}\right) & \text{is stable.} \end{cases}$$

Using Series 8, Task 2 we see that $\begin{bmatrix} \lambda I - A & -B \\ -G & I \end{bmatrix}$ has full column rank anyway.

Thus with Lemma 3.3 we have that $(*)$ is equivalent to $\mathbb{C}_- \supseteq \mathcal{Z}\left(\begin{bmatrix} \lambda I - A & -B \\ -G & I \end{bmatrix}\right) = \mathcal{Z}\left(\begin{bmatrix} I & B \\ & I \end{bmatrix}\begin{bmatrix} \lambda I - (A+BG) & 0 \\ -G & I \end{bmatrix}\right)$

$$\overset{\text{(Homework S.8, T.4)}}{=} \underbrace{\mathcal{Z}(\lambda I - (A+BG))}_{=\sigma(A+BG)} \cup \underbrace{\mathcal{Z}(I)}_{=\emptyset} = \sigma(A+BG)$$

Spectrum = set of Eigenv.

Corollary 3.6: Let $(A,B) \in \mathbb{C}^{n,n} \times \mathbb{C}^{n,m}$ be controllable and let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Then there exists a $G \in \mathbb{C}^{m,n}$ such that $\sigma(A+BG) = \{\lambda_1, \dots, \lambda_n\}$. In particular, by choosing $\lambda_1, \dots, \lambda_n \in \mathbb{C}_-$, this implies that the problem of stabilization via state feedback $(*)$ has a solution if (A,B) is controllable.

Proof: By Corollary 3 from ~~that~~ the handout "Constant controllers for first order systems" there exists $C_1 \in \mathbb{C}^{m,n}$ and invertible $C_2 \in \mathbb{C}^{m,m}$ such that

$$\{\lambda_1, \dots, \lambda_n\} = \mathcal{Z}\left(\begin{bmatrix} \lambda I - A & -B \\ C_1 & C_2 \end{bmatrix}\right) = \mathcal{Z}\left(\begin{bmatrix} I & 0 \\ 0 & C_2^{-1} \end{bmatrix}\begin{bmatrix} \lambda I - A & -B \\ C_1 & C_2 \end{bmatrix}\right)$$

$$= \mathfrak{Z}\left(\begin{bmatrix} \lambda I - A & -B \\ -\underbrace{(-C_2^{-1} C_1)}_{=:\, \mathcal{G}} & I \end{bmatrix}\right) \overset{\text{see above}}{=} \dots \dots = \sigma(A + B\mathcal{G})$$

$\boxtimes$

Reconstruction via observer-synthesis

In the last section we assumed that the whole state $x(t)$ can be measured.

If this is impossible or too expensive one would like to reconstruct the state ~~from~~ from the available information.

For the system

$$(S) \quad \begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \quad , \quad \text{where } A \in \mathbb{C}^{n,n}, B \in \mathbb{C}^{n,m}, C \in \mathbb{C}^{p,n},$$

assume that the available information is $u(t)$ and $y(t)$, i.e., assume that we can only measure the input and output.

If (A, C) is observable we saw in Chapter II (especially Theorem 2.21, 3. and Corollary 2.31) that one can deduce the state x from u and y - at least in theory.

In practice one can make the following ansatz:

Simulate in a computer the associated system

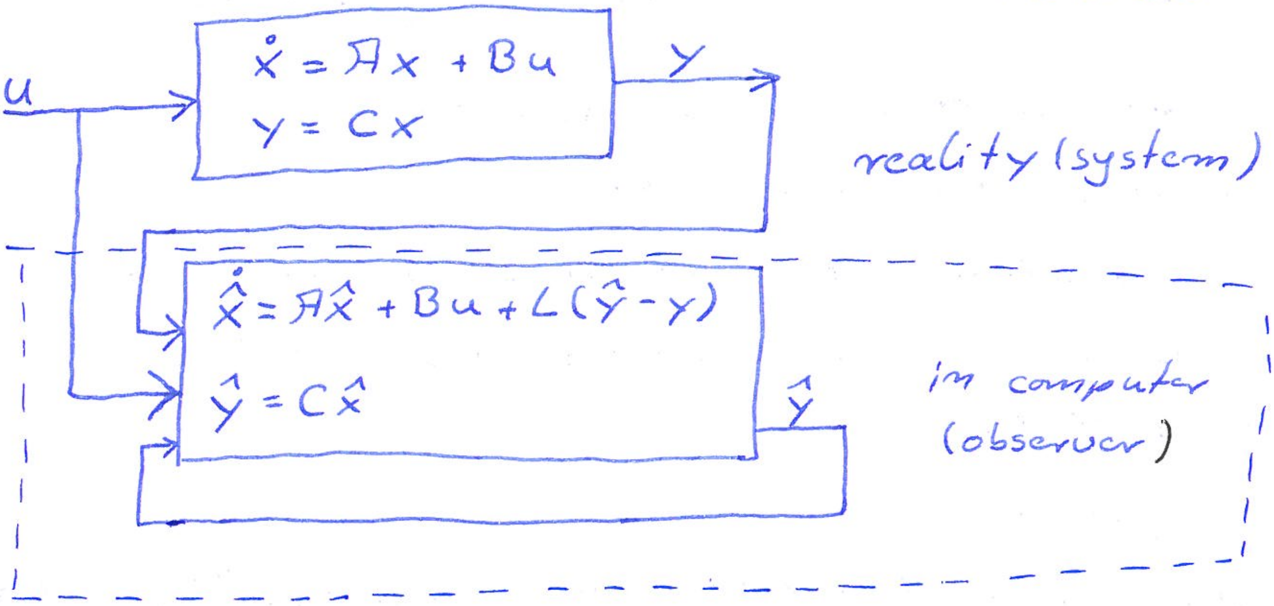
$$(O) \quad \begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L\underbrace{(\hat{y}(t) - y(t))}_{\substack{\text{deviation of the real output } y \\ \text{and estimated output } \hat{y}.}} \\ \hat{y}(t) &= C\hat{x}(t) \end{aligned}$$

where $\hat{x} \in \mathcal{C}_\infty^n$ and $\hat{y} \in \mathcal{C}_\infty^p$ are the estimated state and estimated output, so that we can access $\hat{x}(t)$.

Thereby, we choose $L \in \mathbb{C}^{n,p}$ such that $\hat{x}(t)$ always converges to $x(t)$ for $t \to \infty$, independent of the initial conditions

$$x(0) = x_0 \;,\; \hat{x}(0) = \hat{x}_0 .$$

In this case the system (O) is called an <u>observer</u> for (S).



In other words, we hope that we can reconstruct the state, since we know how the system works, i.e, we know A, B, C.

Combing (S) and (O) we obtain

$$\dot{\hat{x}} = A\hat{x} + Bu + L(\hat{C}\hat{x} - Cx) = (A + LC)\hat{x} + Bu - LCx$$

This implies that the estimated error $e(t) := \hat{x}(t) - x(t)$ admits the law

$$\dot{e}(t) = \dot{\hat{x}}(t) - \dot{x}(t) = (A + LC)\hat{x}(t) + Bu(t) - LCx(t) - Ax(t) - Bu(t)$$

$$= (A + LC)(\hat{x}(t) - x(t)) = (A + LC)e(t).$$

We want that $\lim_{t \to \infty} e(t) = 0$, regardless of the initial conditions in the system (S) and the observer (O):

$$e(0) = \hat{x}(0) - x(0) = \hat{x}_0 - x_0 =: e_0$$

Thus, the problem of reconstruction via observer-synthesis is then to determine an $L \in \mathbb{C}^{p,n}$ such that the behavior of the error $e(t)$

i.e. $(\lambda I - (A+LC))$ is stable.

This is the case if and only if $\sigma(A+LC) \subseteq \mathbb{C}_-$.

Corollary 3.7

Let $(A,C) \in \mathbb{C}^{n,n} \times \mathbb{C}^{p,n}$ be observable and let $\lambda_1, \dots, \lambda_n \in \mathbb{C}$. Then there exists an $L \in \mathbb{C}^{n,p}$ such that

$$\sigma(A+LC) = \{\lambda_1, \dots, \lambda_n\}.$$

In particular, by choosing $\lambda_1, \dots, \lambda_n \in \mathbb{C}_-$, this implies that the problem of reconstruction via observer-synthesis $(*)$ has a solution, if (A,C) is observable.

Proof: If (A,C) is observable then (A^*, C^*) is controllable. Using Corollary 3.6 we obtain the existence of a $G \in \mathbb{C}^{p,n}$ such that $\sigma(A^* + C^* G) = \{\bar{\lambda}_1, \dots, \bar{\lambda}_n\}$.

This implies

$$\sigma(A + \underbrace{G^*}_{L:=} C) = \sigma((A^* + C^* G)^*) = \overline{\sigma(A^* + C^* G)}$$

$$= \overline{\{\bar{\lambda}_1, \dots, \bar{\lambda}_n\}} = \{\lambda_1, \dots, \lambda_n\}$$ and thus the claim $\blacksquare$

Compensators

In this section the results from the two previous sections are combined to develop for the system

$$(S) \quad \begin{aligned} \mathring{x} &= Ax + Bu \\ y &= Cx \end{aligned} \quad , \quad A \in \mathbb{C}^{n,n}, B \in \mathbb{C}^{n,m}, C \in \mathbb{C}^{p,n}$$

a methodology which (theoretically) tells us how to choose the input $u(t)$ such that $x(t) \xrightarrow{t \to \infty} 0$, although we only can measure $y(t)$.

In the section "Stabilization via state-feedback" we computed a $G \in \mathbb{C}^{m,n}$ such that $\sigma(A + BG) \subseteq \mathbb{C}_-$

and then chose the input $u(t) = G x(t)$ $\quad (\square)$

However, since we cannot measure the state $x(t)$ directly, in the section "Reconstruction via observer-synthesis" we computed an $L \in \mathbb{C}^{n,p}$ such that $\sigma(A + L \cdot C) \subseteq \mathbb{C}_-$

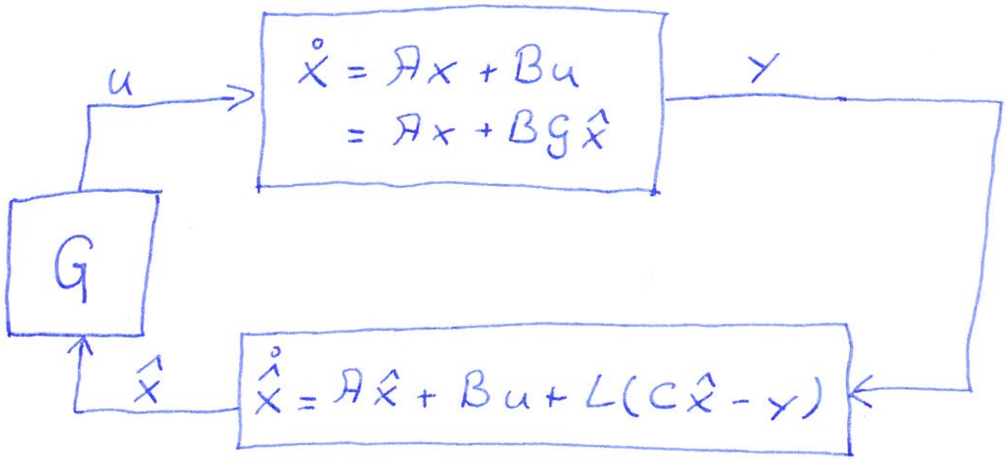
and then imagined to solve

$$(O) \quad \begin{aligned} \mathring{\hat{x}}(t) &= A\hat{x}(t) + Bu(t) + L(\hat{y}(t) - y(t)) \\ \hat{y}(t) &= C x(t) \end{aligned}$$

in a computer to obtain the estimated state $\hat{x}(t) \approx x(t)$. We now use this estimate instead of the unavailable $x(t)$ in $(\square)$: $\quad u(t) = G\hat{x}(t)$

We obtain the equations:

$$\begin{aligned} \mathring{x} &= Ax + Bu = Ax + BG\hat{x} \\ \mathring{\hat{x}} &= A\hat{x} + Bu + L(\hat{y} - y) = A\hat{x} + BG\hat{x} + L(C\hat{x} - Cx) \\ &= (A + BG + LC)\hat{x} - LCx \end{aligned}$$



In matrix form the equations are:

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \underbrace{\begin{bmatrix} A & BG \\ -LC & A+BG+LC \end{bmatrix}}_{=:\mathcal{R}} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}$$

If we can make this system stable (by choosing G and L appropriately) we have found the methodology mentioned in the beginning of this section.

The problem of finding a compensator for (S) is then to

(*) determine $G \in \mathbb{C}^{m,n}$ and $L \in \mathbb{C}^{n,p}$ such that $\sigma(\mathcal{R}) \subseteq \mathbb{C}_-$.

Corollary 3.8

We have $\sigma(\mathcal{R}) = \sigma(A+BG) \cup \sigma(A+LC)$.

In particular, this implies that the problem of finding a compensator has a solution, if (A,B) is controllable and (A,C) is observable.

Proof: With $T := \begin{bmatrix} I & 0 \\ I & I \end{bmatrix}$ we have $T^{-1} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}$

and thus

$$\sigma(\mathcal{A}) = \sigma(T^{-1}\mathcal{A}T) = \sigma\left(\begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}\begin{bmatrix} A+BG & BG \\ A+BG & A+BG+LC \end{bmatrix}\right)$$

$$= \sigma\left(\begin{bmatrix} A+BG & BG \\ 0 & A+LC \end{bmatrix}\right) = \sigma(A+BG) \cup \sigma(A+LC).$$

With Corollary 3.6 and Corollary 3.7 one can then show the solvability of the compensator problem (*).

$\square$

Remark: In reality the measurement of the output (S) $y(t)$, the computation of the solution of (O) $\hat{x}(t)$ in a computer, and the matrix vector product $u(t) = G\hat{x}(t)$ all take time.

This means that the above analysis is unrealistic; especially if beforementioned operations take a lot of time.

It would be more realistic (but also much more complicated) to introduce a delay $\tau > 0$:

$$u(t+\tau) = G\hat{x}(t).$$

The question remains how to compute the G and L such that $\sigma(A+BG) \subseteq \mathbb{C}_-$ and $\sigma(A+LC) \subseteq \mathbb{C}_-$ numerically efficient.

This is answered in the following chapter.