Theorem 4.4

Let $P \in \mathbb{C}[\lambda]^{p,q}$, $H = H^* \in \mathbb{C}^{kq,kq}$, $z_0 \in \mathcal{L}e_+(P)$, and set
$\tilde{H}(\lambda) := \Delta_k^{q\sim}(\lambda) H \Delta_k^q(\lambda)$.

Let $\hat{z} \in \mathcal{L}e_+(P)$ solve (LQ).

Then H is non-negative w.r.t. P and there exists a $\hat{\mu} \in \mathcal{C}_+^p$ such that

$$(\Delta) \qquad \left[\begin{array}{c|c} 0 & P(\frac{d}{dt}) \\ \hline P^\sim(\frac{d}{dt}) & H^\sim(\frac{d}{dt}) \end{array}\right] \begin{bmatrix} \hat{\mu}(t) \\ \hat{z}(t) \end{bmatrix} = 0 \quad t \geq 0.$$

<u>Remarks:</u> Since by (LQ) $\hat{z}(t) = z_0(t)$ for $t \leq 0$ and z_0 does not have to be optimal (Δ) does in general not hold for $t \leq 0$.

<u>Proof:</u> To show non-negativity, let $y \in \mathcal{L}e_+(P)$ with $y(t) = 0$ for $t \leq 0$ be arbitrary. $\Big\}$ $(\square)$

For $s \in \mathbb{R}$ define $z_s(t) := \hat{z}(t) + s y(t) \in \mathcal{L}e_+(P)$
so that for $t \leq 0$ we have $z_s(t) = \underbrace{\hat{z}(t)}_{= z_0 \, (LQ)} + \underbrace{s y(t)}_{= 0 \, (\square)} = z_0(t)$.

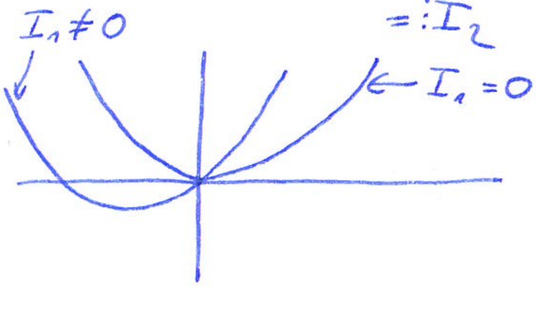
Since $\hat{z}(t)$ is the optimal trajectory (in the sense of (LQ)) z_s cannot be better:

$$\int_0^\infty (\Delta \hat{z})^* H (\Delta \hat{z}) \, dt \leq \int_0^\infty (\Delta_s z_s)^* H (\Delta z_s) \, dt$$

$$= \int_0^\infty (\Delta \hat{z})^* H (\Delta \hat{z}) \, dt + s \underbrace{2 \operatorname{Re} \left\{ \int_0^\infty (\Delta y)^* H (\Delta \hat{z}) \, dt \right\}}_{=: I_1} + s^2 \underbrace{\int_0^\infty (\Delta y)^* H (\Delta y) \, dt}_{=: I_2}$$

$\Rightarrow 0 \leq s I_1 + s^2 I_2$ for all $s \in \mathbb{R}$.

$\Rightarrow I_1 = 0$ and $I_2 \geq 0$.



Thus we have non-negativity $0 \le I_2 = \int_0^\infty (\Delta y)^* H (\Delta y)\, dt$

since y was arbitrary.

Since with y also $i \cdot y \in \mathcal{L}_+(P)$ is a trajectory with

imaginary unit

$i \cdot y(t) = 0$ for $t \le 0$ we obtain that $0 = I_1$ also implies

$$0 = Re\left\{\int_0^\infty \underbrace{(\Delta i y)^*}_{=(i\Delta y)^* = -i(\Delta y)^*} H(\Delta \hat{z})\, dt\right\} = -Re\left\{i \int_0^\infty (\Delta y)^* H (\Delta \hat{z})\, dt\right\}$$

$$= Im\left\{\int_0^\infty (\Delta y)^* H(\Delta \hat{z})\, dt\right\}.$$

This means that the real part and the imaginary part are zero and thus $0 = \int_0^\infty (\Delta y)^* H(\Delta \hat{z})\, dt$

$$= \int_0^\infty \left(\Delta_K^q\left(\tfrac{d}{dt}\right) y\right)^* \left(H \Delta_K^q\left(\tfrac{d}{dt}\right)\hat{z}\right) dt$$

Lemma 1.2

$$\stackrel{\downarrow}{=} \int_0^\infty y^* \left(\Delta_K^{q\sim}\left(\tfrac{d}{dt}\right) H \Delta_K^q\left(\tfrac{d}{dt}\right)\hat{z}\right) dt - \sum_{e=1}^\infty (-1)^e \underbrace{\left(\Delta_K^{q\,\langle e\rangle}\left(\tfrac{d}{dt}\right) y(t)\right)^*}_{=0 \text{ with } (\square)} \cdots$$

$$\cdots \left(H \Delta_K^q\left(\tfrac{d}{dt}\right)\hat{z}(t)\right)\Big|_0^\infty$$

$(*)$

$$= \int_0^\infty y^* \left(\tilde{H}\left(\tfrac{d}{dt}\right)\hat{z}\right) dt \text{ for all } y \in \mathcal{L}_+(P) \text{ with } y(t) = 0,\ t \le 0.$$

Let $r := \operatorname{rank}_{\mathbb{C}(\lambda)} P$ and let $U \in \mathbb{C}[\lambda]^{q,q-r}$ and $V \in \mathbb{C}[\lambda]^{q,r}$ be the kernel and co-kernel spanning matrices, as given by Lemma 1.10, i.e., let a) $[U, V]$ unimodular

b) $PU = 0$

c) PV has full column rank.

Then for all $\alpha \in \mathcal{C}_+^{q-r}$ with $\alpha(t) = 0$, $t \le 0$ we have that

$$U(\tfrac{d}{dt})\,\alpha(t) \overset{b)}{\in} \mathcal{L}_+(P) \quad \left[\text{since } \underbrace{P(\tfrac{d}{dt})U(\tfrac{d}{dt})}_{(PU)(\frac{d}{dt})=0}\alpha(t)=0\right]$$

and $U(\tfrac{d}{dt})\,\alpha(t) = 0 \quad , t \le 0.$

Thus $(*)$ implies $0 = \int_0^\infty (U(\tfrac{d}{dt})\alpha)^* (\tilde{H}(\tfrac{d}{dt})\hat{z})\,dt$

$$\overset{\text{Lemma 4.2}}{=} \int_0^\infty \alpha^* (U(\tfrac{d}{dt})^\sim \tilde{H}(\tfrac{d}{dt})\hat{z})\,dt + \sum_{e=1}^{\infty}(-1)^e \dots \underbrace{\alpha(t)}_{=0} \dots \Big|_0^\infty$$

for all $\alpha \in \mathcal{C}_+^{q-r}$ with $\alpha(t) = 0,\ t \le 0$

$$\Rightarrow U^\sim(\tfrac{d}{dt})\,\tilde{H}(\tfrac{d}{dt})\,\hat{z}(t) = 0 \quad \forall t \ge 0 \qquad (**)$$

Furthermore, by Series 9, Task 4 and using c) there exists a $\hat{\mu} \in \mathcal{C}_+^p$ such that $(PV)(\tfrac{d}{dt})\hat{\mu}(t) = -V^\sim(\tfrac{d}{dt})\,\tilde{H}(\tfrac{d}{dt})\,\hat{z}(t)$.

This implies that for $t \ge 0$ we have

$$P^\sim(\tfrac{d}{dt})\,\hat{\mu}(t) \overset{a)}{=} [U(\tfrac{d}{dt}), V(\tfrac{d}{dt})]^{-\sim}[U(\tfrac{d}{dt}), V(\tfrac{d}{dt})]^\sim P^\sim(\tfrac{d}{dt})\hat{\mu}(t)$$

$$= [U(\tfrac{d}{dt}), V(\tfrac{d}{dt})]^{-\sim}\begin{bmatrix} U^\sim(\tfrac{d}{dt})\ P^\sim(\tfrac{d}{dt})\hat{\mu}(t) \\ V^\sim(\tfrac{d}{dt})\ P^\sim(\tfrac{d}{dt})\hat{\mu}(t)\end{bmatrix}$$

$$= [U(\tfrac{d}{dt}), V(\tfrac{d}{dt})]^{-\sim}\begin{bmatrix} \left.(PU)^\sim(\tfrac{d}{dt})\,\hat{\mu}(t)\right\}\overset{b)}{=} 0 \\ (PV)^\sim(\tfrac{d}{dt})\,\hat{\mu}(t)\end{bmatrix}$$

$$= [U(\tfrac{d}{dt}), V(\tfrac{d}{dt})]^{-\sim}\begin{bmatrix} -U^\sim(\tfrac{d}{dt})\,\tilde{H}(\tfrac{d}{dt})\,\hat{z}(t) \\ -V^\sim(\tfrac{d}{dt})\,\tilde{H}(\tfrac{d}{dt})\,\hat{z}(t)\end{bmatrix}$$

$$= \underbrace{[U(\tfrac{d}{dt}), V(\tfrac{d}{dt})]^{-\sim}[U(\tfrac{d}{dt}), V(\tfrac{d}{dt})]}_{=I}\left(-\tilde{H}(\tfrac{d}{dt})\,\hat{z}(t)\right) = -\tilde{H}(\tfrac{d}{dt})\,\hat{z}(t),$$

which in matrix notation is

$$[P^\sim(\tfrac{d}{dt}), \tilde{H}(\tfrac{d}{dt})]\begin{bmatrix}\hat{\mu}(t) \\ \hat{z}(t)\end{bmatrix} = 0 \quad \text{for } t \ge 0$$

$\blacksquare$

Remark: • Results similar to Theorem 4.3 and 4.4 can be obtained for finite-time optimal control problems of the form

$$\inf_{\substack{z\in\mathcal{L}_*(P)\\ z(t)=z_0(t),\, t\le 0}} \left(\int_0^T (\Delta_K z(t))^* H (\Delta_K z(t))dt + (\Delta_K z(T))^* H (\Delta_K z(T))\right)$$

where $T>0$ is some fixed finite end-time.

• The following theorem states that for (LQ) the past $t<0$ of $z_0(t)$ does not play a role but only the values of $z_0(0)$, $\dot{z}_0(t)$, ...

Theorem 4.5 Let $P\in\mathbb{C}[\lambda]_K^{p,q}$ and $H=H^*\in\mathbb{C}^{Kq,Kq}$. Define $\Theta:\mathbb{C}^{Kq}\to\mathbb{R}\cup\{\infty,-\infty\}$ through

$$\Theta(y):=\inf_{\substack{z\in\mathcal{L}_+(P)\\ \Delta_K z(0)=y}} \int_0^\infty (\Delta_K z(t))^* H (\Delta_K z(t))dt.$$

Then we have

$$\Theta(\Delta_K z_0(0)) = \overbrace{\inf_{\substack{z\in\mathcal{L}_+(P)\\ z(t)=z_0(t),\, t\le 0}} \int_0^\infty (\Delta z(t))^* H (\Delta z(t))dt}^{\text{same as in (LQ)}}$$

for all $z_0\in\mathcal{L}_+(P)$.

Furthermore, if $K=1$ (this implies that $\Delta_K^q(\lambda)=I$ and thus $\Delta_K z=z$) and $P(\lambda)=\lambda F+G$ we also have

$$\Theta(z_0(0)) = \inf_{\substack{z\in\mathcal{L}_+(P)\\ Fz_0(0)=Fz(0)}} \int_0^\infty z^*(t)\, H\, z(t)\, dt$$

Proof: see Appendix.

Para-Hermitian Schur-Form

In this section we use the following notation: we set

$$\mathcal{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \in \mathbb{C}^{2p,2p} \quad \text{and with } M_{11} = M_{11}^*, M_{12}, M_{22} = M_{22}^* \in \mathbb{C}^{p,p}$$

we define the para-Hermitian polynomial

$$\mathcal{N}(\lambda) = \lambda \mathcal{J} + \underbrace{\begin{bmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{22} \end{bmatrix}}_{=: M = M^*}.$$

If $\mathcal{N}$ has no purely imaginary zeros, i.e. $\mathcal{Z}(\mathcal{N}) \cap i\mathbb{R} = \emptyset$ then by Lemma 4.1 the number of zeros in the left and right half plane is the same

$$|\mathcal{Z}(\mathcal{N}) \cap \mathbb{C}_-| = |\mathcal{Z}(\mathcal{N}) \cap \mathbb{C}_+|.$$

In this section we will show that in this case there exists a unitary $V \in \mathbb{C}^{2p,2p}$ such that

$$V^* \mathcal{N}(\lambda) V = \lambda \mathcal{J} + \left[\begin{array}{c|c} K & T \\ \hline T^* & 0 \end{array}\right] = \left[\begin{array}{c|c} K & \lambda I + T \\ \hline -\lambda I + T^* & 0 \end{array}\right],$$

where the (1,2)-block contains all the zeros in the left half plane.

$$\mathcal{Z}(\lambda I + T) = \mathcal{Z}(\mathcal{N}) \cap \mathbb{C}_-$$ and the (2,1)-block contains all zeros in the right half-plane

$$\mathcal{Z}(-\lambda I + T^*) \underset{\text{Lemma 4.1}}{=} -\overline{\mathcal{Z}(\lambda I + T)} = \mathcal{Z}(\mathcal{N}) \cap \mathbb{C}_+.$$

Lemma 4.6: Let $\lambda_0 \in \mathcal{Z}(\mathcal{N})$ be with $Re(\lambda_0) \neq 0$.

Let $x_0 = \begin{bmatrix} v_0 \\ w_0 \end{bmatrix} \in \mathbb{C}^{2p} \setminus \{0\}$ be with $\mathcal{N}(\lambda_0) x_0 = 0$.

(Existence of x_0: Lemma 1.9).

Then there exists a unitary $V \in \mathbb{C}^{2p,2p}$ such that

$$V^* \mathcal{N}(\lambda) V = \lambda \mathcal{J} + \begin{array}{cc} \left[\begin{array}{c:c} * & * \\ \hdashline * & * \end{array}\right] & \begin{array}{c} p \\ p \end{array} \\ \begin{array}{cc} p & p \end{array} & \end{array}$$ (this means that $V^* \mathcal{J} V = \mathcal{J}$ by equating coefficients)

and $V^* x_0 = \beta \cdot \underset{\text{2pth unit vector}}{e_{2p}}$, where $\beta \in \mathbb{C} \setminus \{0\}$.

Proof: [as seen in a lecture of C. Mehl]

From $0 = \mathcal{N}(\lambda_0) x_0 = \lambda_0 \mathcal{J} x_0 + \mathcal{M} x_0$

it follows that $\lambda_0 \underbrace{x_0^* \mathcal{J} x_0}_{\in (i\mathbb{R})} = -x_0^* \mathcal{M} x_0 \in \mathbb{R}$. ($\mathcal{M}$ is hermitian)

$\mathcal{J}$ is skew-Hermitian (Homework, S. 10, T. 4)

Assume to the contrary that $x_0^* \mathcal{J} x_0 \neq 0$. Then

$$\lambda_0 = \frac{1}{\underbrace{x_0^* \mathcal{J} x_0}_{\in (i\mathbb{R})}} x_0^* \mathcal{M} x_0 \in (i\mathbb{R})$$

which contradicts the assumption $Re(\lambda) \neq 0$. Thus

$$0 = x_0^* \mathcal{J} x_0 = x_0^* \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} v_0 \\ w_0 \end{bmatrix} = [v_0^* \; w_0^*] \begin{bmatrix} w_0 \\ -v_0 \end{bmatrix} = v_0^* w_0 - w_0^* v_0$$

$$\Rightarrow v_0^* w_0 = w_0^* v_0 = (v_0^* w_0)^* = \overline{v_0^* w_0} \in \mathbb{R}.$$

Let $P_1 \in \mathbb{C}^{p,p}$ be a unitary Householder-transformation such that

$$P_1 v_0 = \alpha \underset{\text{p-th unit vector}}{e_p}, \ \alpha \in \mathbb{C}$$ (Existence of P_1: Homework S. 10, T. 8)

and set

$$P_1 w_0 := \tilde{w} := \begin{bmatrix} \tilde{w}_1 \\ \vdots \\ \tilde{w}_p \end{bmatrix}$$ to obtain

$$\underbrace{\begin{bmatrix} P_1 & \\ & P_1 \end{bmatrix}}_{=: V_1^*} \begin{bmatrix} v_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \alpha \\ \tilde{w}_1 \\ \vdots \\ \tilde{w}_p \end{bmatrix} \begin{matrix} \}p \\ \\ \}p \end{matrix}$$

which is not yet a multiple of e_{2p} but at least the transformation

$$V_1^* J V_1 = V_1^* \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} P_1^* & 0 \\ 0 & P_1^* \end{bmatrix} = \begin{bmatrix} P_1 & \\ & P_1 \end{bmatrix} \begin{bmatrix} 0 & P_1^* \\ -P_1^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = J$$

leaves J unchanged. Note that

$$\mathbb{R} \ni v_0^* w_0 = v_0^* P_1^* P_1 w_0 = (P_1 v_0)^* (P_1 w_0) = [0, \dots, 0, \bar{\alpha}] \begin{bmatrix} \tilde{w}_1 \\ \vdots \\ \tilde{w}_p \end{bmatrix} = \bar{\alpha} \tilde{w}_p.$$

If $\alpha \neq 0$ we have that

$$\mathbb{R} \ni \frac{\bar{\alpha} \cdot \tilde{w}_p}{|\alpha|^2} = \frac{\bar{\alpha} \cdot \tilde{w}_p}{\bar{\alpha} \cdot \alpha} = \frac{\tilde{w}_p}{\alpha}$$ and thus

$$s := \frac{1}{\sqrt{1 + \left(\frac{\tilde{w}_p}{\alpha}\right)^2}} \in \mathbb{R} \text{ and } c = \frac{\tilde{w}_p}{\alpha} \cdot s \in \mathbb{R}$$

fulfill $c^2 + s^2 = \left(\frac{\tilde{w}_p}{\alpha}\right)^2 s^2 + s^2 = \left(1 + \left(\frac{\tilde{w}_p}{\alpha}\right)^2\right) s^2 = 1$, and

$$\underbrace{\begin{bmatrix} c & -s \\ s & c \end{bmatrix}}_{=: \hat{G}} \begin{bmatrix} \alpha \\ \tilde{w}_p \end{bmatrix} =: \begin{bmatrix} c\alpha - s\tilde{w}_p \\ \hat{w}_p \end{bmatrix} = \begin{bmatrix} s \frac{\tilde{w}_p}{\alpha} \cdot \alpha - s \cdot \tilde{w}_p \\ \hat{w}_p \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \hat{w}_p \end{bmatrix}$$

only $\hat{w}_p$ is defined.

where $\hat{G}$ is an orthogonal (and thus also a unitary) Given rotation.

If $\alpha = 0$ we can set $c = 1$, $s = 0$ and still we have

$c^2 + s^2 = 1$, and $$\underbrace{\begin{bmatrix} c & -s \\ s & c \end{bmatrix}}_{= \hat{G}} \begin{bmatrix} \alpha \\ \tilde{w}_p \end{bmatrix} := \begin{bmatrix} 0 \\ \hat{w}_p \end{bmatrix}$$

by setting $\hat{w}_p := \tilde{w}_p$, where $\hat{G}$ is again unitary.

In both cases this implies that the givens rotation

$$\mathcal{G}^* := \left[\begin{array}{c|c|c|c} I_{p-1} & 0 & 0 & 0 \\ \hline 0 & c & 0 & -s \\ \hline 0 & 0 & I_{p-1} & 0 \\ \hline 0 & s & 0 & c \end{array}\right] \in \mathbb{R}^{2p,2p}$$ is unitary (even orthogonal)

and $$\mathcal{G}^* V_1^* x_0 = \mathcal{G} \begin{bmatrix} \left.\begin{matrix} 0 \\ \vdots \\ 0 \\ \alpha \end{matrix}\right\} p \\ \hline \left.\begin{matrix} \hat{\omega}_1 \\ \vdots \\ \hat{\omega}_p \end{matrix}\right\} p \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \hline \hat{\omega}_1 \\ \vdots \\ \hat{\omega}_{p-1} \\ \hat{\omega}_p \end{bmatrix} =: \begin{bmatrix} 0 \\ \hline \hat{\omega} \end{bmatrix} \begin{matrix} \}p \\ \}p \end{matrix}$$ while still

$$\mathcal{G}^* J \mathcal{G} = \mathcal{G}^* \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \left[\begin{array}{c|c|c|c} I & & 0 & \\ \hline & c & & -s \\ \hline 0 & & I & \\ \hline & s & & c \end{array}\right] = \left[\begin{array}{c|c|c|c} I & & 0 & \\ \hline & c & & s \\ \hline 0 & & I & \\ \hline & -s & & c \end{array}\right] \left[\begin{array}{c|c|c|c} 0 & & I & \\ \hline & s & & c \\ \hline -I & & 0 & \\ \hline & -c & & s \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c} 0 & & I & \\ \hline & cs - sc & & c^2+s^2 \\ \hline -I & & 0 & \\ \hline & -s^2-c^2 & & -sc+cs \end{array}\right] = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} = J.$$

Finally let P_2 again be a unitary Householder-transformation, this time with $P_2 \hat{\omega} = \beta e_p$ so that

$$\underbrace{\begin{bmatrix} P_2 & \\ & P_2 \end{bmatrix}}_{:= V_2^*} \mathcal{G}^* V_1^* x_0 = \begin{bmatrix} 0 \\ \beta e_p \end{bmatrix} = \beta e_{2p}$$ where as before $V_2^* J V_2 = J$.

Thus, setting $V^* := V_2^* \mathcal{G} V_1^* = (V_1 \mathcal{G}^* V_2)^*$ the claim is shown

$\blacksquare$

Theorem 4.7

If the para-Hermitian polynomial matrix

$$\mathcal{N}(\lambda) = \lambda \underbrace{\begin{bmatrix} 0 & I_p \\ -I_p & 0 \end{bmatrix}}_{=: \mathcal{J}_p} + \mathcal{M} \in \mathbb{C}[\lambda]_1^{2p,2p} \text{ has no purely}$$

imaginary zeros, then there exists a unitary $V \in \mathbb{C}^{2p,2p}$ such that

$$V^* \mathcal{N}(\lambda) V = \lambda \mathcal{J}_p + \left[\begin{array}{c|c} K & L \\ \hline L^* & 0 \end{array}\right] = \left[\begin{array}{c|c} K & \lambda I + L \\ \hline -\lambda I + L^* & 0 \end{array}\right]$$

where $K = K^* \in \mathbb{C}^{p,p}$, $L = [\quad] \in \mathbb{C}^{p,p}$ is lower triangular

and $\mathcal{Z}(\lambda I + L) \subseteq \mathbb{C}_-$.

<u>Proof</u>: By induction.

<u>Base case</u>: $(p=1)$ Homework; works in principle exactly as the inductio step. Or for $p=0$ it is trivial.

<u>Inductio-step</u>: Let $\lambda_0 \in \mathcal{Z}(\mathcal{N})$, $\operatorname{Re}(\lambda_0) \neq 0$. Since $\mathcal{N} = \mathcal{N}^\sim$ by Lemma 4.1 also $-\bar{\lambda}_0 \in \mathcal{Z}(\mathcal{N})$ and $\operatorname{Re}(-\bar{\lambda}_0) = -\operatorname{Re}(\lambda_0)$. Thus we can w.l.o.g. assume that $\operatorname{Re}(\lambda_0) < 0$. Let $x_0 \in \mathbb{C}^{2p} \setminus \{0\}$ be with $\mathcal{N}(\lambda_0) x_0 = 0$ (Lemma 1.9). By Lemma 4.6 we obtain the existence of a unitary V with $V^* \mathcal{J}_p V = \mathcal{J}_p$ and

$$V^* x_0 = \beta e_{2p}, \quad \beta \neq 0$$

Then we have

$$0 = V^* \mathcal{N}(\lambda_0) x_0 = \underbrace{V^* \mathcal{N}(\lambda_0) V}_{=:\tilde{\mathcal{N}}(\lambda_0)} V^* x_0 = \tilde{\mathcal{N}}(\lambda_0)\beta e_{2p} \text{ , i.e.,}$$

that the last column of $\tilde{\mathcal{N}}(\lambda_0)$ vanishes. Since (due to the structure of $\mathcal{J}_p = \begin{bmatrix} 0 & I_p \\ -I_p & 0 \end{bmatrix}$) the last column of $\tilde{\mathcal{N}}(\lambda) := V^* \mathcal{N}(\lambda) V$ has the form

$$\tilde{\mathcal{N}}(\lambda) e_{2p} = \begin{bmatrix} \tilde{m}_1 \\ \vdots \\ \tilde{m}_{p-1} \\ \lambda + \tilde{m}_p \\ \tilde{m}_{p+1} \\ \vdots \\ \tilde{m}_{2p} \end{bmatrix}$$ this implies that $\tilde{m}_p = -\lambda_0$ and $\tilde{m}_i = 0$ for $i \neq p$.

Since $\mathcal{M}$ (and thus also $V^* \mathcal{M} V = \tilde{\mathcal{M}}$) is Hermitian the last row of $\tilde{\mathcal{N}}(\lambda)$ has the same / a similar form

$$e_{2p}^* \tilde{\mathcal{N}}(\lambda) = [\overbrace{0, \dots, 0}^{p-1}, -\lambda - \bar{\lambda}_0, \overbrace{0, \dots, 0}^{p}]$$

which means that $\tilde{\mathcal{N}}$ has the form

$$\tilde{\mathcal{N}}(\lambda) = \lambda \left[\begin{array}{c|c|c|c} 0 & & I_{p-1} & \\ \hline & 0 & & 1 \\ \hline -I_{p-1} & & 0 & \\ \hline & -1 & & 0 \end{array}\right] + \begin{array}{cc} \left[\begin{array}{c|c|c|c} \tilde{M}_{11} & \times & \tilde{M}_{12} & 0 \\ \hline \times & \times & \times & -\lambda_0 \\ \hline \tilde{M}_{12}^* & \times & \tilde{M}_{22} & 0 \\ \hline 0 & -\bar{\lambda}_0 & 0 & 0 \end{array}\right] & \begin{array}{l} \}\, p-1 \\ \}\, 1 \\ \}\, p-1 \\ \}\, 1 \end{array} \\ \begin{array}{cccc} \underbrace{\quad}_{p-1} & \underbrace{\quad}_{1} & \underbrace{\quad}_{p-1} & \underbrace{\quad}_{1} \end{array} & \end{array} ,$$

where $\times$ denotes scalars or vector of matching dimension and $\tilde{M}_{11} = \tilde{M}_{11}^*$, $\tilde{M}_{12}$, $\tilde{M}_{22} = \tilde{M}_{22}^* \in \mathbb{C}^{p-1,p-1}$ are constant matrices. Since the submatrix of $\tilde{\mathcal{N}}(\lambda)$ given by

$$\tilde{\mathcal{N}}_1(\lambda) := \lambda \mathcal{J}_{p-1} + \begin{bmatrix} \tilde{M}_{11} & \tilde{M}_{12} \\ \tilde{M}_{12}^* & \tilde{M}_{22} \end{bmatrix} \in \mathbb{C}[\lambda]_1^{2(p-1),2(p-1)}$$

also has purely imaginary eigenvalues

$$\mathcal{Z}(\mathcal{N}) = \mathcal{Z}(\tilde{\mathcal{N}}) = \mathcal{Z}(\tilde{\mathcal{N}}_1) \cup \{\lambda_0, -\bar{\lambda}_0\}$$

(Homework, S. 10, T. 10)

the induction hypothesis implies that there exists a unitary

$$\tilde{V}_1 =: \begin{array}{c} \left[\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline Q_{21} & Q_{22} \end{array}\right] \begin{array}{l} \}\,p-1 \\ \}\,p-1 \end{array} \\ \begin{array}{cc} \underbrace{\qquad}_{p-1} & \underbrace{\qquad}_{p-1} \end{array} \end{array} \in \mathbb{C}^{2(p-1),2(p-1)}$$

such that $\tilde{V}_1^* \tilde{\mathcal{N}}_1(\lambda) \tilde{V}_1 = \lambda \mathcal{J}_{p-1} + \begin{bmatrix} \tilde{K} & \tilde{L} \\ \tilde{L}^* & 0 \end{bmatrix}$ with

$\tilde{L}$ lower triangular. Finally defining

$$\tilde{V} := \left[\begin{array}{c|c|c|c} Q_{11} & & Q_{12} & \\ \hline & 1 & & 0 \\ \hline Q_{21} & & Q_{22} & \\ \hline & 0 & & 1 \end{array}\right] \in \mathbb{C}^{2p,2p}$$ we have

$$\tilde{V}^* \tilde{\mathcal{N}}(\lambda) \tilde{V} = \lambda \mathcal{J}_p + \left[\begin{array}{cc|cc} \tilde{K} & \times & \tilde{L} & 0 \\ \times & \times & \times & -\lambda_0 \\ \hline \tilde{L}^* & \times & 0 & 0 \\ 0 & -\bar{\lambda}_0 & 0 & 0 \end{array}\right] =: \lambda \mathcal{J}_p + \begin{bmatrix} K & L \\ L^* & 0 \end{bmatrix}$$

and thus the claim.

$\blacksquare$