$\mathcal{L}Q$-optimal controller

Although the main Theorem in this section can be formulated in a more general form, we make the following assumption for simplicity:

Assumption 4.8: Let $\lambda F + G \in \mathbb{C}[\lambda]_1^{p,q}$ be such that F has full row rank and $\mathfrak{L}(\lambda F + G)$ is controllable. Let $H \in \mathbb{C}^{q,q}$ be Hermitian positive definite:

$$H = H^* > 0.$$

We need the following Lemma, so that we later are able to apply the results from the previous section.

Lemma 4.9.:

Under the Assumption 4.8 we have that

$$\mathcal{Z}\Bigg(\underbrace{\left[\begin{array}{c|c} 0 & \lambda F + G \\ \hline -\lambda F^* + G^* & H \end{array}\right]}_{=:\,\mathcal{H}}\Bigg) \cap (i\mathbb{R}) = \emptyset \quad \text{, i.e., that}$$

$\mathcal{H} = \mathcal{H}^\sim$ has no purely imaginary ~~eigenvalues~~ zeros.

Proof: Assume to the contrary that there exists a $\lambda_0 \in \mathcal{Z}(\mathcal{H})$ with $\mathrm{Re}(\lambda_0) = 0$. By Lemma 1.9. there exist a vector $\begin{array}{l}\begin{bmatrix} \mu_0 \\ z_0 \end{bmatrix}\begin{array}{l}\}p \\ \}q\end{array}\end{array} \in \mathbb{C}^{p+q} \setminus \{0\}$ such that

$$0 = \left[\begin{array}{c|c} 0 & \lambda_0 F + G \\ \hline -\lambda_0 F^* + G^* & H \end{array}\right] \begin{bmatrix} \mu_0 \\ z_0 \end{bmatrix}$$

$$\Rightarrow 0 = \lambda_0 F z_0 + \mathcal{G} z_0 \quad \text{and} \quad -\lambda_0 F^* \mu_0 + \mathcal{G}^* \mu_0 + H z_0 = 0$$

$$\Rightarrow z_0^* H z_0 = +\lambda_0 z_0^* F^* \mu_0 - \underbrace{z_0^* \mathcal{G}^*}_{=(\mathcal{G} z_0)^* = (-\lambda_0 F z_0)^*} \mu_0$$

$$= \lambda_0 z_0^* F^* \mu_0 + \bar{\lambda}_0 z_0^* F^* \mu_0$$

$$= 2 \underbrace{\mathrm{Re}(\lambda_0)}_{=0} z_0^* F^* \mu_0 = 0$$

$$\overset{H>0}{\Rightarrow} z_0 = 0 \Rightarrow 0 = (-\lambda F^* + \mathcal{G}^*)\mu_0 . \qquad (*)$$

Since F has full row rank, Series 3, Task 9 implies that

$$\mathrm{rank}_{\mathbb{C}(\lambda)} (\lambda F + \mathcal{G}) = p .$$

Since $\mathfrak{L}_e(\lambda F + \mathcal{G})$ is controllable Theorem 2.13 implies that $\mathcal{Z}(\lambda F + \mathcal{G}) = \emptyset$.

Thus, Theorem 1.13 implies that $(\lambda F + \mathcal{G})$ is left prime and thus $(\lambda F + \mathcal{G})^\sim = (-\lambda F^* + \mathcal{G}^*)$ is right prime (Series 5, Task 7).

This, especially means that $(-\lambda_0 F^* + \mathcal{G}^*)$ has full column rank and thus $(*)$ shows that $\mu_0 = 0$. This contradicts $\begin{bmatrix} \mu_0 \\ z_0 \end{bmatrix} \neq 0$.

$\blacksquare$

In the following Theorem we will show the existence of controllers that enforce the optimal behavior (in the sense of (RQ)) on the system.

Theorem 4.10

Under Assumption 4.8 there exists a $Z \in \mathbb{C}^{p,q}$ such that

1) There exists a $C \in \mathbb{C}^{q-p,q}$ such that

$$\begin{bmatrix} I & 0 \\ Z^* & I \end{bmatrix} \left[\begin{array}{c|c} 0 & \lambda F + G \\ \hline -\lambda F^* + G^* & H \end{array}\right] \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} = \left[\begin{array}{c|c} 0 & \lambda F + G \\ \hline -\lambda F^* + G^* & C^* \cdot C \end{array}\right]$$

2) Every C as in 1) is a regular, stabilizing controller for $\mathcal{L}(\lambda F + G)$ which fulfills

a) $\mathcal{Z}(\mathcal{M}) \cap \mathbb{C}_- = \mathcal{Z}\left(\begin{bmatrix} \lambda F + G \\ C \end{bmatrix}\right)$

b) $\mathcal{L}_+(\mathcal{M}) = \begin{bmatrix} Z \\ I \end{bmatrix} \mathcal{L}_+\left(\begin{bmatrix} \lambda F + G \\ C \end{bmatrix}\right)$

c) $\operatorname{rank} \begin{bmatrix} F \\ C \end{bmatrix} = q.$

Proof: Since F has full row rank there exists an invertible V_1 such that

$F V_1 = [I_p, 0]$. Partition $G \cdot V_1 =: [\underbrace{-A}_{p}, \underbrace{-B}_{q-p}]$ and

$$V_1^* H V_1 = \begin{bmatrix} Q & S \\ S^* & R \end{bmatrix}, \quad Q = Q^* > 0, \quad R = R^* > 0, \text{ accordingly:}$$

$H > 0$

Set $X_1 := \left[\begin{array}{c|c} I_p & \\ \hline & V_1 \end{array}\right]$ and $X_2 := \begin{bmatrix} I_p & 0 & 0 \\ 0 & I_p & 0 \\ R^{-1} B^* & -R^{-1} S^* & I_{q-p} \end{bmatrix}$ so that

$$\mathcal{X}_2^* \mathcal{X}_1^* \mathcal{M}(\lambda) \mathcal{X}_1 \mathcal{X}_2 = \mathcal{X}_2^* \left[\begin{array}{c|c} 0 & \lambda(FV_1)+(GV_1) \\ \hline -\lambda(V_1^* F^*)+(V_1^* G^*) & V_1^* H V_1 \end{array}\right] \mathcal{X}_2$$

$$= \mathcal{X}_2^* \begin{bmatrix} 0 & \lambda I - A & -B \\ \lambda I - A^* & Q & S \\ -B^* & S^* & R \end{bmatrix} \begin{bmatrix} I_p & 0 & 0 \\ 0 & I_p & 0 \\ R^{-1}B^* & -R^{-1}S^* & I_{q-p} \end{bmatrix}$$

$$= \begin{bmatrix} I_p & 0 & BR^{-1} \\ 0 & I_p & -SR^{-1} \\ 0 & 0 & I_{q-p} \end{bmatrix} \left[\begin{array}{c|c|c} \underbrace{-BR^{-1}B^*}_{=:\mathcal{M}_{11}} & \lambda I \underbrace{-A+BR^{-1}S^*}_{=:\mathcal{M}_{12}} & -B \\ \hline -\lambda I \underbrace{-A^*+SR^{-1}B^*}_{=:\mathcal{M}_{21}=\mathcal{M}_{12}^*} & \underbrace{Q-SR^{-1}S^*}_{=:\mathcal{M}_{22}} & S \\ \hline 0 & 0 & R \end{array}\right]$$

$$= \left[\begin{array}{cc|c} \mathcal{M}_{11} & \lambda I + \mathcal{M}_{12} & 0 \\ -\lambda I + \mathcal{M}_{12}^* & \mathcal{M}_{22} & 0 \\ \hline 0 & 0 & R \end{array}\right].$$

By Lemma 4.9 we have that

$$\mathcal{Z}(\mathcal{M}) = \mathcal{Z}\left(\begin{bmatrix} \mathcal{M}_{11} & \lambda I + \mathcal{M}_{12} \\ -\lambda I + \mathcal{M}_{12}^* & \mathcal{M}_{22} \end{bmatrix}\right) \cup \underbrace{\mathcal{Z}(R)}_{=\emptyset}$$

has no purely imaginary zeros. Thus, we can use the para Hermitian Schur form (Theorem 4.7), to deduce the existence of a unitary $V_3 \in \mathbb{C}^{2p,2p}$ such that with $\mathcal{X}_3 := \left[\begin{array}{c|c} V_3 & \\ \hline & I_{q-p} \end{array}\right]$ and $\mathcal{X}_1 \mathcal{X}_2 \mathcal{X}_3 =: \tilde{\mathcal{Y}} =: \begin{array}{cc} \left[\begin{array}{c|c} \mathcal{Y}_1 & \tilde{\mathcal{Y}}_2 \\ \hline \mathcal{Y}_3 & \tilde{\mathcal{Y}}_4 \end{array}\right] & \begin{array}{c} p \\ q \end{array} \\ \begin{array}{cc} p & q \end{array} & \end{array}$

we have

$$\tilde{\mathcal{Y}}^* \mathcal{M}(\lambda) \tilde{\mathcal{Y}} = \mathcal{X}_3^* (\mathcal{X}_2^* \mathcal{X}_1^* \mathcal{M}(\lambda) \mathcal{X}_1 \mathcal{X}_2) \mathcal{X}_3 = \left[\begin{array}{cc|c} K & \lambda I + L & 0 \\ -\lambda I + L^* & 0 & 0 \\ \hline 0 & 0 & R \end{array}\right].$$

(α)

Let $W \in \mathbb{C}^{q,q}$ be unitary such that $\mathcal{Y}_4 := \tilde{\mathcal{Y}}_4 \cdot W$ is Hermitian: $\mathcal{Y}_4 = \mathcal{Y}_4^*$ (β) (Existence of W: Homework)

and set $\mathcal{Y}_2 := \tilde{\mathcal{Y}}_2 W$ so that

$$\mathcal{Y} := \tilde{\mathcal{Y}} \left[\begin{array}{c|c} I_p & \\ \hline & W \end{array}\right] = \begin{bmatrix} \mathcal{Y}_1 & \mathcal{Y}_2 \\ \mathcal{Y}_3 & \mathcal{Y}_4 \end{bmatrix} \text{ and}$$

$$\mathcal{Y}^* \mathcal{DL}(\lambda) \begin{bmatrix} \mathcal{Y}_1 & \mathcal{Y}_2 \\ \mathcal{Y}_3 & \mathcal{Y}_4 \end{bmatrix} = \left[\begin{array}{c|c} K & (\lambda[I,0] + [L,0])W \\ \hline W^*\left(-\lambda \begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} L^* \\ 0 \end{bmatrix}\right) & W^* \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} W \end{array}\right]$$

$$=: \left[\begin{array}{c|c} K & \lambda\hat{F} + \hat{g} \\ \hline -\lambda\hat{F}^* + \hat{g} & \hat{H} \end{array}\right]. \quad (\gamma)$$

To show that $\mathcal{Y}_4$ is invertible assume to the contrary that there exists a $U \in \mathbb{C}^{q,e}$, $e>0$ with orthonormal columns $U^*U = I$ such that $\operatorname{kernel}(\mathcal{Y}_4) = \operatorname{image}(U)$. (σ)

From the second block column in (γ) we obtain

$$\begin{bmatrix} \lambda\hat{F} + \hat{g} \\ \hat{H} \end{bmatrix} U = \mathcal{Y}^* \begin{bmatrix} 0 & \lambda F + g \\ -\lambda F^* g^* & H \end{bmatrix} \underbrace{\begin{bmatrix} \mathcal{Y}_2 \\ \mathcal{Y}_4 \end{bmatrix} U}_{= \begin{bmatrix} \mathcal{Y}_2 U \\ \mathcal{Y}_4 U \end{bmatrix} = \begin{bmatrix} \mathcal{Y}_2 U \\ 0 \end{bmatrix}} \quad (\varepsilon)$$

$$= \begin{bmatrix} \mathcal{Y}_1^* & \mathcal{Y}_3^* \\ \mathcal{Y}_2^* & \mathcal{Y}_4^* \end{bmatrix} \begin{bmatrix} 0 \\ (-\lambda F^* + g^*)\mathcal{Y}_2 U \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{Y}_3^* \\ \mathcal{Y}_4^* \end{bmatrix} (-\lambda F^* + g^*) \mathcal{Y}_2 U$$

which implies (by only considering the lower block equation in (ε))

$$U^*\hat{H}U = \underbrace{U^* Y_4^*}_{=(Y_4 U)^* \overset{(\delta)}{=} 0}(-\lambda F^* + G^*) Y_2 U = 0.$$

Since $0 \leq \hat{H}$ there exists an $\hat{L}$ such that $\hat{L}^*\hat{L} = H$ which means that for every $y \in \mathbb{C}^e$ we have

$$0 = y^* U^* \hat{H} U y = \|\hat{L} U y\|_2^2$$

$$\Rightarrow \hat{L} U y = 0 \Rightarrow \hat{L}^* \hat{L} U y = 0 \Rightarrow \hat{H} U y = 0 \Rightarrow \hat{H} U = 0.$$

This gives 2nd block equation of

$$0 = \hat{H} U \overset{(\varepsilon)}{=} Y_4^*(-\lambda F^* + G^*) Y_2 U \overset{(\beta)}{=} \lambda(-Y_4 F^* Y_2 U) + Y_4 G^* Y_2 U$$

or in other words

$$0 = -Y_4 F^* Y_2 U \quad \text{and} \quad 0 = Y_4 G^* Y_2 U$$

$$\Rightarrow \text{image}(-F^* Y_2 U) \subseteq \text{Kernel } Y_4 \overset{(\delta)}{=} \text{image}(U)$$

$$\Rightarrow \text{image}(G^* Y_2 U) \subseteq \text{Kernel } Y_4 \overset{(\delta)}{=} \text{image}(U).$$

Thus there exist $\tilde{E}, \tilde{A} \in \mathbb{C}^{e,e}$ such that

$$-F^* Y_2 U = U\tilde{E} \quad \text{and} \quad G^* Y_2 U = U\tilde{A}. \qquad (\xi)$$

Since $Y \begin{bmatrix} 0 \\ I \end{bmatrix} = \begin{bmatrix} Y_2 \\ Y_4 \end{bmatrix}$ and U both have full column rank.

also $\begin{bmatrix} Y_2 \\ Y_4 \end{bmatrix} U = \begin{bmatrix} Y_2 U \\ 0 \end{bmatrix}$ and thus also

$Y_2 U$ has full column rank. (η)

Since F is assumed to have full row rank we have that F^* has full column rank and thus (by (η)) also $F^* Y_2 U$ has full column rank.

Then from

$$\ell = \operatorname{rank}(F^* Y_2 U) \overset{(\mathcal{S})}{=} \operatorname{rank}(U\tilde{E}) \le \operatorname{rank}(\tilde{E}) \overset{\text{rank} \,\le\, \text{number of cols}}{\le} \ell$$

we find that $\tilde{E}$ is invertible.

Since $\mathcal{Z}(\lambda\tilde{E} + \tilde{A}) = \mathcal{Z}(\tilde{E}(\lambda I + \tilde{E}^{-1}\tilde{A}))$

$$= \mathcal{Z}(\lambda I + \tilde{E}^{-1}\tilde{A}) = \sigma(-\tilde{E}^{-1}\tilde{A}) \overset{\ell > 0}{\neq} \emptyset$$

there exists a $\lambda_0 \in \mathcal{Z}(\lambda\tilde{E} + \tilde{A})$ and (with Lemma 1.9) a $v_0 \in \mathbb{C}^\ell \setminus \{0\}$ such that $0 = (\lambda_0 \tilde{E} + \tilde{A}) v_0$

$$\Rightarrow 0 = U(\lambda_0\tilde{E} + \tilde{A}) v_0 \overset{(\mathcal{S})}{=} (-\lambda_0 F^* + G^*) \underbrace{Y_2 U v_0}_{=: w_0 \neq 0}.$$

This means $\operatorname{rank}(-\lambda_0 F^* + G^*) < p \overset{F^* \text{ has full column rank}}{=} \operatorname{rank}_{\mathbb{C}(\lambda)} (-\lambda F^* + G^*)$ (η)

$$\overset{\text{Lemma 1.9.}}{\Rightarrow} \lambda_0 \in \mathcal{Z}(-\lambda F^* + G^*) \overset{\text{Lemma 4.1}}{=} \overline{-\mathcal{Z}(\lambda F + G)} \overset{\text{Theorem 2.13, since } \mathcal{L}(\lambda F + G) \text{ is controllable}}{=} \emptyset \quad \lightning$$

$\Rightarrow Y_4$ is invertible.

This implies that also

$$\underbrace{\begin{bmatrix} Y_1 & Y_2 \\ Y_3 & Y_4 \end{bmatrix}}_{= Y} \begin{bmatrix} I & 0 \\ -Y_4^{-1} Y_3 & Y_4^{-1} \end{bmatrix} = \left[\begin{array}{c|c} Y_1 - Y_2 Y_4^{-1} Y_3 & Y_2 Y_4^{-1} \\ \hline 0 & I \end{array}\right]$$

(Schur complement)

is invertible, and thus $(Y_1 - Y_2 Y_4^{-1} Y_3)$ is invertible, and thus setting

$$\hat{Y} := \left[\begin{array}{c|c} I & 0 \\ \hline -Y_4^{-1} Y_3 & Y_4^{-1} \end{array}\right] \left[\begin{array}{c|c} (Y_1 - Y_2 Y_4^{-1} Y_3)^{-1} & 0 \\ \hline 0 & I \end{array}\right]$$

$$= \begin{bmatrix} * & 0 \\ * & Y_4^{-1} \end{bmatrix} =: \left[\begin{array}{c|c} \hat{Y}_1 & 0 \\ \hline \hat{Y}_3 & \hat{Y}_4 \end{array}\right] \qquad (\nu)$$

we have

$$\mathcal{Y}\hat{\mathcal{Y}} = \left[\begin{array}{c|c} I & \mathcal{Y}_2\mathcal{Y}_4^{-1} \\ 0 & I \end{array}\right] =: \left[\begin{array}{c|c} I & Z \\ 0 & I \end{array}\right] \quad \text{and}$$

$$\begin{bmatrix} I & 0 \\ Z^* & I \end{bmatrix} \mathcal{M}(\lambda) \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} = \hat{\mathcal{Y}}^*(\mathcal{Y}^*\mathcal{M}(\lambda)\mathcal{Y})\hat{\mathcal{Y}}$$

$$\overset{(6)}{=} \hat{\mathcal{Y}}^* \begin{bmatrix} K & \lambda\hat{F}+\hat{G} \\ -\lambda\hat{F}+\hat{G} & \hat{H} \end{bmatrix} \begin{bmatrix} \hat{\mathcal{Y}}_1 & 0 \\ \hat{\mathcal{Y}}_3 & \hat{\mathcal{Y}}_4 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{\mathcal{Y}}_1^* & \hat{\mathcal{Y}}_3^* \\ 0 & \hat{\mathcal{Y}}_4^* \end{bmatrix} \begin{bmatrix} * & \lambda\hat{F}\hat{\mathcal{Y}}_4 + \hat{G}\hat{\mathcal{Y}}_4 \\ * & \hat{H}\hat{\mathcal{Y}}_4 \end{bmatrix}$$

$$= \begin{bmatrix} * & * \\ * & \hat{\mathcal{Y}}_4^*\hat{H}\hat{\mathcal{Y}}_4 \end{bmatrix} =: \begin{bmatrix} \lambda\tilde{S}+\tilde{K} & \lambda\tilde{F}+\tilde{G} \\ -\lambda\tilde{F}^*+\tilde{G}^* & \hat{\mathcal{Y}}_4^*\hat{H}\hat{\mathcal{Y}}_4 \end{bmatrix} \quad (7)$$

where $\tilde{S}=-\tilde{S}^*$ and $\tilde{K}=\tilde{K}^*$, since the matrix polynomial (7) is still para-Hermitian.

Looking only at the coefficient of λ in (7) we find

$$\begin{bmatrix} I & 0 \\ Z^* & I \end{bmatrix} \begin{bmatrix} 0 & F \\ -F^* & 0 \end{bmatrix} \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} = \begin{bmatrix} \tilde{S} & \tilde{F} \\ -\tilde{F}^* & 0 \end{bmatrix}$$

$\Vert$

$$\begin{bmatrix} I & 0 \\ Z^* & I \end{bmatrix} \begin{bmatrix} 0 & F \\ -F^* & -F^*Z \end{bmatrix} = \begin{bmatrix} 0 & F \\ -F^* & Z^*F-F^*Z \end{bmatrix}$$

$\Rightarrow \tilde{S} = 0$, $\tilde{F} = F$, $0 = ZF^* - F^*Z$

and looking only at the constant term in (τ) we find

$$\begin{bmatrix} I & 0 \\ Z^* & I \end{bmatrix} \begin{bmatrix} 0 & g \\ g^* & H \end{bmatrix} \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} = \begin{bmatrix} \tilde{K} & \tilde{g} \\ \tilde{g}^* & \hat{Y}_4^* \hat{H} \hat{Y}_4 \end{bmatrix}$$

$$\|$$

$$\begin{bmatrix} I & 0 \\ Z^* & I \end{bmatrix} \begin{bmatrix} 0 & g \\ g^* & H + g^* Z \end{bmatrix} = \begin{bmatrix} 0 & g \\ g^* & H + g^* Z + Z^* g \end{bmatrix}$$

$\Rightarrow \tilde{K} = 0$, $\tilde{g} = g$, $\hat{Y}_4^* \hat{H} \hat{Y}_4 = H + g^* Z + Z^* g$

Computing a Cholesky factorization of $0 < R = L^* L$ we find

$$\hat{Y}_4^* \hat{H} \hat{Y}_4 \overset{(\gamma)}{=} \hat{Y}_4^* W^* \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} W \hat{Y}_4 = \hat{Y}_4^* W^* \begin{bmatrix} 0 \\ L \end{bmatrix} \underbrace{[L \;\; 0] W \hat{Y}_4}_{=: C}$$

$= C^* C \quad (K)$

and thus (τ) reads

$$\begin{bmatrix} I & 0 \\ Z^* & I \end{bmatrix} \mathcal{M}(\lambda) \begin{bmatrix} I & Z \\ 0 & I \end{bmatrix} = \left[\begin{array}{c|c} 0 & \lambda F + g \\ \hline -\lambda F^* + g^* & C^* C \end{array}\right]$$

which means that 1.) is shown.

For 2.) Let $C \in \mathbb{C}^{q-p,q}$ be any matrix with $\hat{Y}_4^* \hat{H} \hat{Y}_4 = C^* C$.

Let $D \in \mathbb{C}^{p,q}$ be such that $[C^*, D^*]$ is invertible

(C has full row rank, since $\operatorname{rank} \hat{Y}_4^* \hat{H} \hat{Y}_4 = \operatorname{rank} \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} = q - p$)

and observe that

$$\underbrace{\begin{bmatrix} I & \\ & [C^*, D^*] \end{bmatrix}}_{=:C_1} \begin{bmatrix} \lambda F + G \\ \begin{bmatrix} C \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \lambda F + G \\ C^* C \end{bmatrix} = \begin{bmatrix} \lambda F + G \\ \hat{Y}_4^* \hat{H} \hat{Y}_4 \end{bmatrix} \qquad (\lambda)$$

second block column in (c), with $\hat{F} = F$, $\hat{G} = G$

$$= \hat{Y}^* \begin{bmatrix} \lambda \hat{F} + \hat{G} \\ \hat{H} \end{bmatrix} \hat{Y}_4 \overset{\text{second block column in } (\gamma)}{=} \hat{Y}^* \begin{bmatrix} [\lambda I + L, 0] W \\ W^* \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} W \end{bmatrix} \hat{Y}_4$$

$$\underbrace{\hat{Y}^* \begin{bmatrix} I & \\ & W^* \end{bmatrix}}_{=:W_1} \begin{bmatrix} \lambda I + L & 0 \\ 0 & 0 \\ 0 & R \end{bmatrix} \underbrace{W \hat{Y}_4}_{=:W_2}$$

where C_1, W_1, W_2 are invertible.

<u>C is regular</u>: We have

$$\operatorname{rank} C = \operatorname{rank} C^* C = \operatorname{rank} W^{-*} \hat{Y}_4^{-*} C^* C \hat{Y}_4^{-1} W^{-1}$$

$$\overset{(K)}{=} \operatorname{rank} \begin{bmatrix} 0 & 0 \\ 0 & R \end{bmatrix} = \operatorname{rank} \overset{R > 0}{R} = q - p \quad \text{and}$$

$$\operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} \lambda F + G \\ C \end{bmatrix} \overset{(\lambda)}{=} \operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} \lambda I + L & 0 \\ 0 & R \end{bmatrix} = p + (q - p)$$

$$\overset{F \text{ has full row rank}}{\leq} \operatorname{rank}_{\mathbb{C}(\lambda)} (\lambda F + G) + \operatorname{rank} C$$

<u>C is stabilizing</u>: We have

$$\operatorname{rank}_{\mathbb{C}(\lambda)} \begin{bmatrix} \lambda F + G \\ C \end{bmatrix} = \text{see "regular"} = p + (q - p) = q, \text{ i.e., that}$$

$\mathfrak{L}\left(\begin{bmatrix} \lambda F + G \\ C \end{bmatrix}\right)$ is autonomous and

$$\mathcal{Z}\left(\begin{bmatrix} \lambda F + g \\ C \end{bmatrix}\right) \overset{(\lambda)}{=} \mathcal{Z}\left(\begin{bmatrix} \lambda I + L & 0 \\ 0 & R \end{bmatrix}\right)$$

$$= \underbrace{\mathcal{Z}(\lambda I + L)}_{\subseteq \mathbb{C}_- \text{ by Theorem 4.7}} \cup \underbrace{\mathcal{Z}(R)}_{=\emptyset} \subseteq \mathbb{C}_- .$$

a): We have

$$\mathcal{Z}\left(\begin{bmatrix} \lambda F + g \\ C \end{bmatrix}\right) = \text{see "stabilizing"} = \underbrace{\mathcal{Z}(\lambda I + L)}_{\subseteq \mathbb{C}_-}$$

$$= \Big[\mathcal{Z}(\lambda I + L) \cup \underbrace{-\mathcal{Z}(\lambda I + L)}_{\subseteq \mathbb{C}_+}\Big] \cap \mathbb{C}_-$$

$$\overset{\text{Lemma 4.1}}{=} \Big[\mathcal{Z}(\lambda I + L) \cup \mathcal{Z}(-\lambda I + L^*)\Big] \cap \mathbb{C}_-$$

$$= \mathcal{Z}\left(\begin{bmatrix} K & \lambda I + L & 0 \\ -\lambda I + L^* & 0 & 0 \\ 0 & 0 & R \end{bmatrix}\right) \cap \mathbb{C}_-$$

$$\overset{(\alpha)}{=} \mathcal{Z}(\gamma \mathcal{Y}) \cap \mathbb{C}_- .$$

b): We have

$$\mathcal{L}_+(\gamma \mathcal{Y}) \overset{(\alpha)}{=} \mathcal{L}_+\left(\tilde{\mathcal{Y}}^{-*} \begin{bmatrix} \dots \end{bmatrix} \tilde{\mathcal{Y}}^{-1}\right) = \tilde{\mathcal{Y}} \mathcal{L}_+\left(\begin{bmatrix} K & -\lambda I + L & 0 \\ -\lambda I + L^* & 0 & 0 \\ 0 & 0 & R \end{bmatrix}\right)$$

$$= \tilde{\mathcal{Y}} \left\{ \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \;\middle|\; z_1 \in \mathcal{L}_+(-\lambda I + L^*) \overset{\text{since } \mathcal{Z}(-\lambda I + L^*) \subseteq \mathbb{C}_+}{=} \{0\},\ \left(\tfrac{d}{dt} I + L\right) z_2 + K z_1 = 0,\ z_3 \in \mathcal{L}_+(R) \underset{R>0}{=} \{0\} \right\}$$

$$= \tilde{Y}\left\{\begin{bmatrix}0\\ z_2\\ 0\end{bmatrix} \;\middle|\; z_2 \in \mathcal{L}_+(\lambda I + L)\right\}$$

$$= \tilde{Y}\left\{\begin{bmatrix}0\\ \tilde{z}_2\end{bmatrix} \;\middle|\; \tilde{z}_2 \in \mathcal{L}_+\left(\begin{bmatrix}\lambda I + L & 0\\ 0 & R\end{bmatrix}\right)\right\}$$

$$= \underbrace{\tilde{Y}\begin{bmatrix}0\\ I\end{bmatrix}}_{= \begin{bmatrix}Y_1 & \tilde{Y}_2\\ Y_3 & \tilde{Y}_4\end{bmatrix}\begin{bmatrix}0\\ I\end{bmatrix}} \mathcal{L}_+\left(\begin{bmatrix}\lambda I + L & 0\\ 0 & R\end{bmatrix}\right) \overset{(a)}{=} \begin{bmatrix}\tilde{Y}_2\\ \tilde{Y}_4\end{bmatrix} \mathcal{L}_+\left(W_1^{-1} C_1 \begin{bmatrix}\lambda F + G\\ C\\ 0\end{bmatrix} W_2^{-1}\right)$$

$$= \underbrace{\begin{bmatrix}\hat{Y}_2\\ \hat{Y}_4\end{bmatrix} W_2} \mathcal{L}_+\left(\begin{bmatrix}\lambda F + G\\ C\end{bmatrix}\right) = \begin{bmatrix}Z\\ I\end{bmatrix} \mathcal{L}_+\left(\begin{bmatrix}\lambda F + G\\ C\end{bmatrix}\right).$$

$$\overset{(a)}{=} \begin{bmatrix}\tilde{Y}_2\\ \tilde{Y}_4\end{bmatrix} W \hat{Y}_4 \overset{(R)}{=} \begin{bmatrix}Y_2\\ Y_4\end{bmatrix} \hat{Y}_4 \overset{(2e)}{=} \begin{bmatrix}Y_2\\ Y_4\end{bmatrix} Y_4^{-1}$$

$$= \begin{bmatrix}Y_2 Y_4^{-1}\\ I\end{bmatrix} \overset{(2e)}{=} \begin{bmatrix}Z\\ I\end{bmatrix} \cancel{= \begin{bmatrix}Z\\ I\end{bmatrix} \mathcal{L}_+\left(\begin{bmatrix}\lambda F + G\\ C\end{bmatrix}\right)}$$

$\blacksquare$