Technische Universität Berlin Summer term 2014
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Control Theory

4. Exercise

(Discussion on June 10, 2014)

Exercise 4.1: (Kronecker product)

(a) Let W, X, Y, Z be matrices of appropriate dimensions such that the products WX and YZ are defined. Show that $(W \otimes Y)(X \otimes Z) = (WX) \otimes (YZ)$.

(b) Let S, G be invertible matrices. Show that also $S \otimes G$ is invertible and that $(S \otimes G)^{-1} = S^{-1} \otimes G^{-1}$.

(c) Let $A \in R^{n,n}$ and $B \in \mathbb{R}^{m,m}$. Further, let A have the eigenvalues $\lambda_1, \ldots, \lambda_n$ and B have the eigenvalues $\mu_1, \ldots, \mu_m$. Show that

$$\sigma(A \otimes B) = \{\lambda_i \mu_j \,|\, i = 1, \ldots, n,\; j = 1, \ldots, m\}.$$

Exercise 4.2: (Sylvester equation)
Let $A \in \mathbb{R}^{n,n}$, $B \in \mathbb{R}^{m,m}$ and $W \in \mathbb{R}^{n,m}$. Show that the Sylvester equation $AX + XB = W$ has a solution X if and only if the matrices

$$M_1 = \begin{bmatrix} A & W \\ 0 & -B \end{bmatrix}, \quad M_2 = \begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix}$$

are similar.

Exercise 4.3: (Partial Stabilization using the sign function method)
Let $(A, B) \in \mathbb{R}^{n,n} \times \mathbb{R}^{n,m}$ and assume that $\sigma(A) = \Lambda_- \cup \Lambda_+ := \{\lambda_1, \ldots, \lambda_k\} \cup \{\lambda_{k+1}, \ldots, \lambda_n\}$ with $Re(\lambda_j) < 0$ for $j = 1, \ldots, k$ and $Re(\lambda_j) > 0$ for $j = k+1, \ldots, n$. Let

$$A = S \begin{bmatrix} J_- & 0 \\ 0 & J_+ \end{bmatrix} S^{-1}, \quad \sigma(J_-) = \Lambda_-,\; \sigma(J_+) = \Lambda_+,$$

be the Jordan canonical form of A. Then

$$sign(A) := S \begin{bmatrix} -I_k & 0 \\ 0 & I_{n-k} \end{bmatrix} S^{-1}.$$

(a) Show that $P := \frac{1}{2}(I - sign(A))$ is a projector onto $\mathcal{S}_-$ (the A-invariant subspace corresponding to Λ_-).

(b) Let $G \in \mathbb{R}^{n,n}$, $H \in \mathbb{R}^{m,m}$, $W \in \mathbb{R}^{n,m}$ with $\sigma(G), \sigma(H) \subset \mathbb{C}^+$ and consider the Sylvester equation

$$GX + XH = W. \tag{1}$$

Show that

$$sign\left(\begin{bmatrix} G & W \\ 0 & -H \end{bmatrix}\right) = \begin{bmatrix} I_n & 2X \\ 0 & -I_m \end{bmatrix},$$

whereby X is the solution of the Sylvester equation (1).