Technische Universität Berlin Summer term 2014
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Control Theory

6. Exercise
(Discussion on July 7, 2014)

Exercise 6.1: (Hamiltonian matrices)
Let $\mathcal{H} \in \mathbb{R}^{2n,2n}$ and $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. Show that the following statements are equivalent:

(i) The matrix $\mathcal{H}$ is Hamiltonian, i.e., $\mathcal{H}^T J + J\mathcal{H} = 0$.

(ii) The matrix $J\mathcal{H}$ is symmetric.

(iii) There exist $F, G, H \in \mathbb{R}^{n,n}$ with $G = G^T$ and $H = H^T$, such that $\mathcal{H} = \begin{bmatrix} F & G \\ H & -F^T \end{bmatrix}$.

(iv) The matrix $\mathcal{H}$ is skew-adjoint with respect to the inner product

$$\langle x, y \rangle_J := y^T J x, \quad x, y \in \mathbb{R}^{2n},$$

i.e., for all $x, y \in \mathbb{R}^{2n}$ it holds that $\langle \mathcal{H}x, y \rangle_J = -\langle x, \mathcal{H}y \rangle_J$.

Exercise 6.2: (Isotropic invariant subspaces)
Let $\mathcal{H} \in \mathbb{R}^{2n,2n}$ be Hamiltonian and let

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \in \mathbb{R}^{2n,r}$$

with $X_1, X_2 \in \mathbb{R}^{n,r}$, such that the columns of X span an $\mathcal{H}$-invariant subspace $\mathcal{U}$, i.e.,

$$\mathcal{H}X = XZ$$

for a matrix $Z \in \mathbb{R}^{r,r}$.
Show: If the eigenvalues corresponding to $\mathcal{U}$ are all in the left open half plane, i.e., $\sigma(Z) \subset \mathbb{C}^-$, then $\mathcal{U}$ is *isotropic*, i.e., for all $x, y \in \mathcal{U}$ it holds that $\langle x, y \rangle_J = 0$.
Hint: Lyapunov theory.

Exercise 6.3: (Hamiltonian and symplectic matrices)
Let $\mathbb{H}_{2n}$ be the set of Hamiltonian matrices and let $\mathbb{S}_{2n}$ be the set of symplectic matrices. A matrix $\mathcal{S} \in \mathbb{R}^{2n,2n}$ is called symplectic, if $\mathcal{S}$ is orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle_J$, i.e., for all $x, y \in \mathbb{R}^{2n}$ it holds that $\langle \mathcal{S}x, \mathcal{S}y \rangle_J = \langle x, y \rangle_J$. Show:

(i) The matrix $\mathcal{S}$ is symplectic if and only if $\mathcal{S}^T J \mathcal{S} = J$.

(ii) It holds $|\det(\mathcal{S})| = 1$.

(iii) The set of symplectic matrices $\mathbb{S}_{2n}$ together with the matrix multiplication forms a group.

(iv) $\mathbb{H}_{2n}$ is the Lie-Algebra corresponding to the Lie group $\mathbb{S}_{2n}$, i.e., every $\mathcal{H} \in \mathbb{H}_{2n}$ solves

$$\frac{d}{dt}\Big(\langle (I+t\mathcal{H})x, (I+t\mathcal{H})y\rangle_J = \langle x, y\rangle_J\Big)\Big|_{t=0} \quad \text{for all } x, y \in \mathbb{R}^{2n}$$

and $\mathbb{H}_{2n}$ is a (non-associative) algebra with respect to the product

$$[\mathcal{H}_1, \mathcal{H}_2] := \mathcal{H}_1\mathcal{H}_2 - \mathcal{H}_2\mathcal{H}_1, \quad \mathcal{H}_1, \mathcal{H}_2 \in \mathbb{H}_{2n}$$

that satisfies the following properties:

a) $[\mathcal{H}_1, \mathcal{H}_2] \in \mathbb{H}_{2n}$ for all $\mathcal{H}_1, \mathcal{H}_2 \in \mathbb{H}_{2n}$.

b) $[\mathcal{H}, \mathcal{H}] = 0$ for all $\mathcal{H} \in \mathbb{H}_{2n}$.

c) $[\mathcal{H}_1 + \mathcal{H}_2, \mathcal{H}_3] = [\mathcal{H}_1, \mathcal{H}_3] + [\mathcal{H}_2, \mathcal{H}_3]$ for all $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 \in \mathbb{H}_{2n}$.

d) $[[\mathcal{H}_1, \mathcal{H}_2], \mathcal{H}_3] + [[\mathcal{H}_2, \mathcal{H}_3], \mathcal{H}_1] + [[\mathcal{H}_3, \mathcal{H}_1], \mathcal{H}_2] = 0$ for all $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3 \in \mathbb{H}_{2n}$. (Jacobi identity)

(v) It holds that $\mathcal{S}^{-1}\mathcal{H}\mathcal{S} \in \mathbb{H}_{2n}$ for all $\mathcal{S} \in \mathbb{S}_{2n}$, $\mathcal{H} \in \mathbb{H}_{2n}$.