Technische Universität Berlin
Fakultät II – Institut für Mathematik
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Summer 2017
Hand in solutions: Wed, April 26
Discussion in class: Wed, April 26

Kontrolltheorie – Control theory

1. Exercise

– Preliminaries –

Each group (max. 2 students) hands in the solution of **one** exercise (max. 3 points).

1.1) fundamental solutions, adjoint equation (3 points)

a) Prove Lemma 1.8:
The fundamental solution $\Phi(t,s)$ satisfies for all $r,s,t \in \mathbb{R}$ the following properties:

i) $\Phi(t,s) = \Phi(t,r)\Phi(r,s)$ (semigroup property),

ii) $\Phi(t,s)$ is invertible and $\Phi(t,s)^{-1} = \Phi(s,t)$,

iii) $\frac{\mathrm{d}}{\mathrm{d}s}\Phi(t,s) = -\Phi(t,s)A(s)$,

iv) $x(t) := \Phi(t,t_0)x_0$ is the unique solution of the IVP $\dot{x}(t) = A(t)x(t)$ with $x(t_0) = x_0$.

b) Prove Lemma 2.15:
Let $\Phi(t,t_0)$ be the fundamental solution of $\dot{x}(t) = A(t)x(t)$. Then, $\Phi(t_0,t)^T$ is the fundamental solution of the adjoint equation $\dot{z}(t) = -A(t)^T z(t)$.

1.2) invariant subspaces (3 points)

a) Prove Lemma 1.22:
Consider matrices $A \in \mathbb{R}^{n,n}$, $X \in \mathbb{R}^{n,k}$ and $\mathcal{V} := \operatorname{Bild} X$. Then, $\mathcal{V}$ is A-invariant, if and only if there exists a matrix $B \in \mathbb{R}^{k,k}$ such that $AX = XB$. If X has full column rank, then it holds in addition $\Lambda(B) \subseteq \Lambda(A)$.

b) Prove Lemma 1.24:
Let $\mathcal{V} = \operatorname{Bild} X \subseteq \mathbb{R}^n$ be an A-invariant subspace. Then, $\mathcal{V}$ equals the span of certain (generalized) eigenvectors of A.

1.3) example: parabolic antenna (3 points)

We consider the (simplified) example of a parabolic antenna, given by

$$\dot{\varphi}(t) = \omega(t), \qquad j\dot{\omega}(t) = -r\omega(t) + ku(t)$$

with initial condition $\varphi(0) = 0$ and $\omega(0) = \omega_0$. The aim is to control the system such that the antenna is continually directed to a satellite or spacecraft. The variables are given by the rotation angle φ and the rotational velocity ω. By j we denote the moment of inertia of the engine that causes the rotation, r includes friction, and k is an amplification factor. The control $u(t)$ equals the input voltage that drives the motor.

Find the solution of the system for the three following cases of the control function:

a) $u(t) = 0$ (free system),

b) $u(t) = \alpha\omega(t)$, $\alpha \in \mathbb{R}$,

c) $u(t) = e^{-t}$.

Furthermore, find an input u such that $\varphi(1) = \pi$. For this, use the ansatz $\omega(t) = \omega_0 + \gamma t$.