Technische Universität Berlin
Fakultät II – Institut für Mathematik
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Summer 2017
Hand in solutions: Wed, May 10
Discussion in class: Wed, May 10

Kontrolltheorie – Control theory

2. Exercise

– Controllability –

Each group (max. 2 students) hands in the solution of **the assigned** exercise (max. 3 points).

2.1) invariant subspaces (again), kernel and range of $W(t_0, t_1)$ (3 points)

Groups: 1, 4, 7, 10, 13, 16, 19

a) Prove Lemma 1.24:
Let $\mathcal{V} = \operatorname{Bild} X \subseteq \mathbb{R}^n$ be an A-invariant subspace. Then, $\mathcal{V}$ equals the span of certain (generalized) eigenvectors of A.

b) Prove Theorem 2.13:
For the controllability Gramian $W(t_0, t_1)$ and every $z \in \mathbb{R}^n$ it holds that

i) $\mathcal{C}(t_0, t_1, 0) = \operatorname{Range} W(t_0, t_1)$
$= \{x_0 \in \mathbb{R}^n \mid \text{there exists } u \in \mathcal{U}_{\text{ad}} \text{ such that } x_0 = -\int_{t_0}^{t_1} \Phi(t_0, s) B(s) u(s) \, ds\}$,

ii) $W(t_0, t_1) z = 0 \;\Leftrightarrow\; B(t)^T \Phi(t_0, t)^T z \equiv 0$ on $[t_0, t_1]$, i.e.,
$\operatorname{Ker} W(t_0, t_1) = \{z \in \mathbb{R}^n \mid B(t)^T \Phi(t_0, t)^T z \equiv 0 \text{ on } [t_0, t_1]\}$.

2.2) controllability Gramian I (3 points)

Groups: 2, 5, 8, 11, 14, 17

a) Prove Theorem 2.17, iii) ⇒ i):
If for each $t_0 \in \mathbb{R}$ there exists a $t_1 > t_0$ such that $W(t_0, t_1)$ is positive definite, then the LTV system is controllable.

b) Prove Remark 2.19:
The controllability Gramian $W(t_0, t_1)$ is positive definite, if and only if $\hat{W}(t_0, t_1)$ is positive definite,

$$\hat{W}(t_0, t_1) := \int_{t_0}^{t_1} \Phi(t_1, s) B(s) B(s)^T \Phi(t_1, s)^T \, ds.$$

2.3) controllability Gramian II, examples (3 points)

Groups: 3, 6, 9, 12, 15, 18

a) Prove Lemma 2.20:
A LTI system with asymptotically stable matrix $A \in \mathbb{R}^{n,n}$ is controllable, if and only if the matrix W is positive definite,

$$W := \int_0^\infty e^{As} B B^T e^{A^T s} \, ds.$$

b) Which of the following systems are controllable? In both cases we have $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $n > 1$,

$$\text{i)} \quad \dot{x} = x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u, \qquad \text{ii)} \quad \dot{x} = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \dots & \dots & -\alpha_{n-1} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u.$$