Kontrolltheorie – Control theory

6. Exercise

– LQ optimal control –

Each group (max. 2 students) hands in the solution of **the assigned** exercise (max. 3 points).

6.1) representation theorem for $t_* = \infty$ (3 points)

Groups: 13-19

Prove Theorem 4.5:
Consider $A, Q \in \mathbb{R}^{n,n}$, $B, S \in \mathbb{R}^{n,m}$, and $R \in \mathbb{R}^{m,m}$. Further, let Q, R be symmetric and

$$R > 0, \qquad \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \geq 0.$$

We define $\hat{P} := BR^{-1}B^T$, $\hat{A} := A - BR^{-1}S^T$, $\hat{Q} := Q - SR^{-1}S^T$, and W as a symmetric solution of the algebraic Riccati equation such that $F := R^{-1}(S^T + B^T W)$ stabilizes (A, B). Then, the minimum of

$$J(x,u) = \frac{1}{2} \int_0^\infty \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \mathrm{d}t,$$

constrained by $\dot{x} = Ax + Bu$ with $x(0) = x_0$, is attained for the feedback law $u(t) := -Fx(t)$. The minimum is given by $\frac{1}{2} x_0^T W x_0$.

6.2) spectrum of Hamiltonian matrices (3 points)

Groups: 7-12

a) For $\hat{P} := BR^{-1}B^T$ and $\hat{A} := A - BR^{-1}S^T$ prove that

$$(A, B) \text{ stabilizable } \Leftrightarrow (\hat{A}, B) \text{ stabilizable } \Leftrightarrow (\hat{A}, \hat{P}) \text{ stabilizable.}$$

Hint: Show for the second part that B and $\hat{P}$ have the same range.

b) Prove Lemma 4.10:
Let $\mathcal{H} \in \mathbb{R}^{2n,2n}$ be Hamiltonian. Then, $\mathcal{H}$ is similar to $-\mathcal{H}$. Furthermore, the spectrum of $\mathcal{H}$ is symmetric w.r.t. the real and imaginary axis.

6.3) orthogonal-symplectic matrices (3 points)

Groups: 1-6

An orthogonal matrix $Q \in \mathbb{R}^{2n,2n}$ is called *orthogonal-symplectic* if there exist $U, V \in \mathbb{R}^{n,n}$ such that

$$Q = \begin{bmatrix} U & -V \\ V & U \end{bmatrix}.$$

Prove Lemma 4.20:
The set of orthogonal-symplectic matrices forms a multiplicative group (w.r.t. standard matrix multiplication). Furthermore, a transformation with an orthogonal-symplectic matrix Q preserves the property of being Hamiltonian, i.e., for a Hamiltonian matrix $\mathcal{H}$ also $Q^T \mathcal{H} Q$ is Hamiltonian.