Linear and Integer Programming (ADM II)

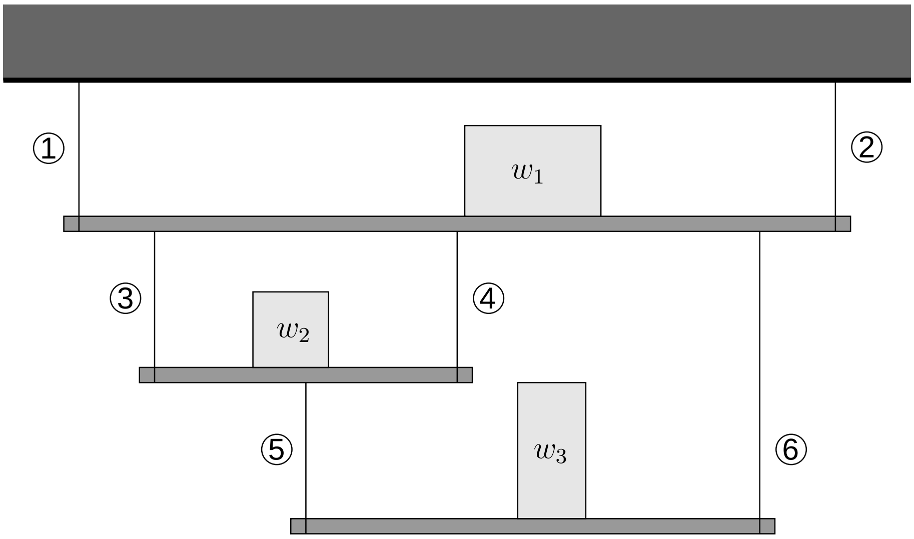
Martin Skutella
Axel Werner
Torsten Ueckerdt
Jannik Matuschke

Exercise sheet 2

Deadline: Mon, 29 Oct 2007, 12:15 in MA 041

Exercise 2. **11 points**

Consider the following scaffolding that consists of three platforms connected in the shown way, hanging at a fixed ceiling.



The connections ① to ⑥ are realised by wires. There has to be at least one wire at each connection, but there can also be more, in which case the carrying capacity of the connection increases linearly. However, every platform must have the same number of wires on both sides. The weight capacity of one wire is 50kg, independent of its length. For simplicity we ignore the weight of the wires themselves. Each connection has a length of 1m, except connection ⑥, which is 2m long. There is a total of 50m of wire available to implement all connections. Finally, there is the weight of the three platforms, which is 10kg each.

The problem is to maximize the total weight that can be loaded onto the construction. For stability reasons it has to be ensured that the weight on the upper platform does not exceed the sum of the weights on the two other platforms; accordingly, the weight on the lower platform may also not exceed the sum of the weights on the others. We ignore the deeper physics of the construction and assume that it makes no difference where the weights are positioned on the respective platforms.

please turn over

a) Set up a linear program that models the problem.
(*Remark:* It might turn out that some of the variables are required to be integer... for now ignore this additional constraint and think about it again towards the end of the semester...)

b) Rewrite the above LP in standard form. What would you expect to be the dimension of the feasible set?

c) Give a basic feasible solution (not necessarily optimal) and its associated basis.

Exercise 3. **6 points**

Consider the following linear optimization problem (after you convinced yourself that it is indeed a linear program):

$$\begin{array}{ll} \text{maximize} & x_n \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} |x_i| \leq 1. \end{array}$$

a) Sketch the feasible set for $n = 3$ and give

   (i) a basic solution that is *not* feasible,

   (ii) a basic feasible solution that is *not* optimal,

   (iii) the optimal solution.

b) Show that for $n \geq 3$ the optimal solution is degenerate. How many active constraints are there for the optimal solution?

Exercise 4. **8 points**

Consider a linear optimization problem over the standard form polyhedron $P = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\}$. Suppose that the matrix $\mathbf{A}$ has dimension $m \times n$ and that its rows are linearly independent. For each of the following statements, decide whether it is true or false and give a proof or a counterexample, respectively.

a) If $n = m + 1$, then P has at most two basic feasible solutions.

b) The set of all optimal solutions is bounded.

c) At every optimal solution, no more than m variables can be positive.

d) If there is more than one optimal solution, then there exist at least two basic feasible solutions that are optimal.

Exercise 5. **8 points**

For each of the following problems either reformulate it as a linear program or explain why this may be not possible.

PLEASE TURN OVER

a) With $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c}_i \in \mathbb{R}^n$, $d_i \in \mathbb{R}$ for $i = 1, \ldots, k$,

$$\begin{array}{ll} \text{maximize} & \min\limits_{i=1,\ldots,k} (\mathbf{c}_i^\top \mathbf{x} + d_i) \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{1}. \end{array}$$

b) With $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{c}_i \in \mathbb{R}^n$, $d_i \in \mathbb{R}$ for $i = 1, \ldots, k$,

$$\begin{array}{ll} \text{minimize} & \min\limits_{i=1,\ldots,k} (\mathbf{c}_i^\top \mathbf{x} + d_i) \\ \text{subject to} & \mathbf{A}\mathbf{x} \geq \mathbf{b}. \end{array}$$

c) With $c_1, \ldots, c_n \in \mathbb{R}$,

$$\begin{array}{ll} \text{minimize} & \sum\limits_{i=1,\ldots,n} c_i |x_i| \\ \text{subject to} & \mathbf{A}\mathbf{x} \geq \mathbf{b}. \end{array}$$

d)

$$\begin{array}{ll} \text{Minimize} & 2x_1 + 3|x_2 - 10| \\ \text{subject to} & |x_1 + 2| + |x_2| \leq 5. \end{array}$$

Exercise 6. **(Tutorial session)**

Let $s, t \in \mathbb{R}$. Find necessary and sufficient conditions on s and t such that the linear program

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & s x_1 + t x_2 \leq 1 \\ & x_1, x_2 \geq 0 \end{array}$$

a) has at least one optimal solution,

b) has exactly one optimal solution,

c) has no solution,

d) is unbounded.

Exercise 7. **(Tutorial session)**

For each of the following statements give either a proof or a counterexample.

a) Every convex function $\mathbb{R} \to \mathbb{R}$ is monotone.

b) Every convex function $\mathbb{R} \to \mathbb{R}$ is continuous.

c) Every convex function $\mathbb{R} \to \mathbb{R}$ is differentiable.

d) The sum of two convex functions $\mathbb{R} \to \mathbb{R}$ is convex.

e) The product of two convex functions $\mathbb{R} \to \mathbb{R}$ is convex.

f) The minimum of k different convex functions $\mathbb{R} \to \mathbb{R}$ is convex.