Technische Universität Berlin

Institut für Mathematik
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# Linear and Integer Programming (ADM II)

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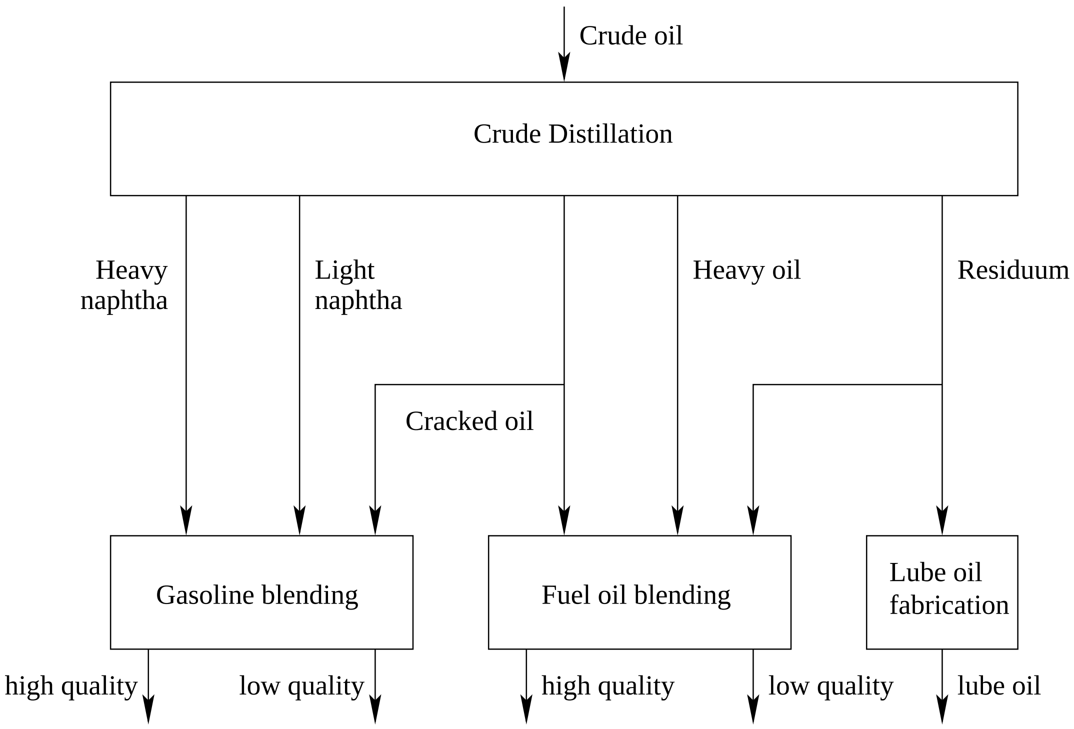
## Exercise sheet 3

Deadline: Wed, 7 Nov 2007, 12:15 in MA 042

**Exercise 8.** **10 points**

Consider the following simplified model of producing different types of gasoline and fuel oil from crude oil.




Crude oil is transformed by a number of distillation processes into different kinds of oil, which is then blended into different kinds of marketable products.

During distillation of each barrel of crude oil, 14% is transformed into heavy naphtha, 25% into light naphta, 21% into cracked oil, and 16% into heavy oil, while the remaining 24% of residuum can still be used to produce lube oil. The distillation process costs money: producing 1 barrel of heavy or light naphta costs \$0.40, of cracked oil \$1.10, of heavy oil \$0.30 and collecting 1 barrel of residuum costs \$0.20.

please turn over

For the blending process certain quality constraints have to be met. High quality gasoline has to consist of at least 70% of heavy naphtha and at least 20% of cracked oil. High quality fuel oil may not contain any residuum, while the low quality may contain at most 15% of the residuum. For marketing reasons the refinery decides to produce at least half as many of the low quality gasoline as of the high quality gas.

Finally, the selling prices per barrel are $365 for the high quality gasoline, $260 for the low quality, $200 for high quality fuel oil, and $140 for the low quality fuel oil. The lube oil still earns $45 per barrel. One barrel of crude oil costs the refinery $90 and for the planning period 1000 barrels will be delivered.

Naturally, the refinery wants to maximise its profit. Set up a linear program that models the problem and solve it with `CPLEX`.

*Remarks (to think about):* What happens if you drop the prerequisite of the 1000 barrels of crude oil? Why? How much lube oil is produced? Why?

**Exercise 9.** **5 points**

Consider the problem of minimizing $\mathbf{c}^\top \mathbf{x}$ over a polyhedron $P$. A vector $\mathbf{d}$ is a *feasible direction* at the point $\mathbf{x} \in P$ if there is some $\theta > 0$ such that $\mathbf{x} + \theta \mathbf{d} \in P$. Prove:

a) A feasible solution $\mathbf{x}$ is optimal if and only if $\mathbf{c}^\top \mathbf{d} \geq 0$ for every feasible direction $\mathbf{d}$ at $\mathbf{x}$.

b) A feasible solution $\mathbf{x}$ is the unique optimal solution if and only if $\mathbf{c}^\top \mathbf{d} > 0$ for every nonzero feasible direction $\mathbf{d}$ at $\mathbf{x}$.

**Exercise 10.** **5 points**

Consider a feasible solution $\mathbf{x}$ to a standard form problem, and let $Z := \{i \mid x_i = 0\}$. Show that $\mathbf{x}$ is an optimal solution if and only if the linear programming problem

$$\begin{array}{rl} \text{minimize} & \mathbf{c}^\top \mathbf{d} \\ \text{subject to} & \mathbf{A}\mathbf{d} = \mathbf{0} \\ & d_i \geq 0, \quad \forall i \in Z, \end{array}$$

has an optimal cost of zero.

**Exercise 11.** **12 points**

a) Consider the polyhedron decribed by the following constraints.

$$\begin{array}{rcrcrcr} -3x_1 & & & + & 2x_3 & \leq & -2 \\ 3x_1 & + & 6x_2 & + & 2x_3 & \leq & 10 \\ 12x_1 & + & 3x_2 & + & x_3 & \leq & 26 \\ -6x_1 & - & 9x_2 & + & 7x_3 & \leq & 2 \\ x_1 & - & x_2 & - & 2x_3 & \leq & 8 \\ 2x_1 & + & x_2 & - & 2x_3 & \leq & 20 \end{array}$$

Show that the points $(-8, 10, -13)^\top$ and $(2, 0, 2)^\top$ are both degenerate basic feasible solutions. In either case explain the respective reason for the degeneracy and whether you can remove it without changing the feasible set.

b) For the polyhedron described by

$$\begin{array}{rcrcrcrl} x_1 & & & + & 3x_3 & - & 2x_4 & \leq 1 \\ & & 2x_2 & - & 5x_3 & & & \leq 1 \\ & & & & x_3 & + & x_4 & \leq 0 \\ & & & & x_3 & - & x_4 & \leq 0 \\ & & & & \multicolumn{3}{r}{x_1, x_2, x_3, x_4} & \geq 0 \end{array}$$

show that all basic feasible solutions are degenerate. Rewrite the constraints to remove degeneracy and list the adjacent basic feasible solutions.

**Exercise 12.** **(Tutorial session)**

Give either a proof or a counterexample for the following statement: There exist $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ such that the two problems

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array} \qquad \text{and} \qquad \begin{array}{ll} \text{minimize} & -\mathbf{c}^\top \mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

are both unbounded.

**Exercise 13.** **(Tutorial session)**

Consider the linear optimization problem

$$\begin{array}{ll} \text{minimize} & x_1 + 2x_2 + x_3 \\ \text{subject to} & \begin{array}{rcrcrcr} -6x_1 & + & 9x_2 & + & x_3 & \leq & 21 \\ -2x_1 & - & x_2 & - & x_3 & \leq & 3 \\ & & -3x_2 & - & x_3 & \leq & 3 \\ 9x_1 & + & 6x_2 & + & 5x_3 & \leq & 27 \\ x_1 & - & x_2 & & & \leq & 3 \end{array} \end{array}$$

Show that the feasible set is unbounded. Can you still find an optimal solution?