Technische Universität Berlin | Institut für Mathematik
Wintersemester 07/08

Linear and Integer Programming (ADM II)

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Exercise sheet 4

Deadline: Wed, 14 Nov 2007, 12:15 in MA 042

Exercise 14. **10 (+3) points**

Suppose n people start at the same place at time 0 to travel to a destination that is D miles away. They can walk and besides that they have one bicycle at their disposal that carries one person at a time. The walking and bicycling speed of person j is given by w_j and b_j, respectively. The task is to find the earliest point in time after which all n persons have arrived at the destination.

a) Show that the linear program

$$\begin{array}{lrcll} \text{minimize} & t & & & \\ \text{subject to} & t - x_j - x'_j - y_j - y'_j & \geq & 0 & (j = 1, \dots, n) \\ & t - \sum_{j=1}^{n} y_j - \sum_{j=1}^{n} y'_j & \geq & 0 & \\ & w_j x_j - w_j x'_j + b_j y_j - b_j y'_j & = & D & (j = 1, \dots, n) \\ & \sum_{j=1}^{n} b_j y_j - \sum_{j=1}^{n} b_j y'_j & \leq & D & \\ & x_j, x'_j, y_j, y'_j & \geq & 0 & (j = 1, \dots, n) \end{array}$$

provides a lower bound on the best possible arrival time of the last person.

b) Write a `ZIMPL` file and use `CPLEX` to find lower bounds for the arrival time for the following values:

	D	n	$w_1, \dots, w_n$	$b_1, \dots, b_n$
(i)	10	3	$2, 4, 2$	$12, 16, 12$
(ii)	45	9	$3, 2, 4, 3, 3, 5, 4, 2, 6$	$14, 11, 15, 13, 17, 17, 14, 10, 18$
(iii)	100	20	$w_j = (10 + j)/5$	$b_j = (20 + j)/2$

c) (**Optional**) Give an example for which the optimum of the above linear program is a strictly smaller value than the arrival time of the last person.

Exercise 15. **6 points**

Let $\mathbf{x}$ be a basic feasible solution associated with some basis matrix $\mathbf{B}$. Prove:

a) If the reduced cost of every nonbasic variable is positive, then $\mathbf{x}$ is the unique optimal solution.

b) If $\mathbf{x}$ is the unique optimal solution and is nondegenerate, then the reduced cost of every nonbasic variable is positive.

Exercise 16. **12 points**

Consider the problem

$$\begin{array}{lrcrcl} \text{minimize} & -2x_1 - x_2 & & & & \\ \text{subject to} & x_1 & - & 2x_2 & \leq & 2 \\ & x_1 & - & x_2 & \leq & 2 \\ & x_1 & + & x_2 & \leq & 6 \\ & -4x_1 & + & x_2 & \leq & 1 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

a) Convert the problem into standard form and construct a basic feasible solution at which $(x_1, x_2) = (0, 0)$.

b) Carry out the simplex algorithm, starting with the basic feasible solution of part a). At each iteration list all basic directions and indicate those that are feasible directions.

c) Draw a graphical representation of the problem in terms of the original variables x_1, x_2, and indicate the path taken by the simplex algorithm, as well as the basic feasible directions in each iteration.

Exercise 17. **8 points**

Consider the simplex method applied to a standard form problem and assume that the rows of the matrix $\mathbf{A}$ are linearly independent. For each of the statements below, give either a proof or a counterexample.

a) A variable that has just left the basis cannot reenter in the very next iteration.

b) A variable that has just entered the basis cannot leave in the very next iteration.

c) If there had been an unconstrained variable x_j that was replaced by two variables $x_j^+, x_j^- \geq 0$ to obtain standard form, then in every iteration at most one of the variables x_j^+, x_j^- is nonzero.

d) Let $\mathbf{x}^*$ be an optimal basic feasible solution and B a basis for $\mathbf{x}^*$. If there is a "second best" basic feasible solution $\tilde{\mathbf{x}}$ (i.e. we have $\mathbf{c}^\top\mathbf{x} > \mathbf{c}^\top\tilde{\mathbf{x}} > \mathbf{c}^\top\mathbf{x}^*$ for every basic feasible solution $\mathbf{x} \neq \tilde{\mathbf{x}}, \mathbf{x}^*$), then $\tilde{\mathbf{x}}$ can be obtained from B by exchanging exactly one basic variable.