Technische Universität Berlin

Institut für Mathematik

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# Linear and Integer Programming (ADM II)

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## Exercise sheet 5

Deadline: Wed, 21 Nov 2007, 12:15 in MA 042

**Exercise 18.** **10 points**

For the construction of a new bridge over the Tay a financing plan has to be established. Table 1 (left) gives the estimated cost over the 6 years of construction.

The city of Dundee plans to raise the funds needed to pay these costs by issuing bonds. Such a bond is valid up to 6 years. It can be taken out every 1st of January and is due on the 31st December of the year that it is due—the validity period is fixed beforehand. Of course, interest has to be paid on bonds when they are due, depending on how long they are valid, see Table 1 (right).

Money that is not used for construction can be invested at the Royal Bank of Scotland at an interest rate of 6.8% anually.

The problem is to find out how many bonds to which terms should be issued each year to keep the outstanding debts at the end as low as possible.

a) Set up a linear program that models the problem.

| year | cost |
|---|---|
| 1 | 20 Mio £ |
| 2 | 17 Mio £ |
| 3 | 23 Mio £ |
| 4 | 24 Mio £ |
| 5 | 25 Mio £ |
| 6 | 21 Mio £ |

| length of validity | overall interest rate |
|---|---|
| 1 years | 7% |
| 2 years | 15% |
| 3 years | 23% |
| 4 years | 32% |
| 5 years | 41% |
| 6 years | 50% |

Table 1: Construction costs each year and interest rates for bonds.

b) Use ZIMPL and CPLEX to find an optimal solution.

c) Suppose that all bonds issued are due to the end of the 6th year. Adapt the above linear program to this special situation and find the optimal solution?

**Exercise 19.** **7 points**

Solve the following linear program with the revised two-phase simplex method:

$$\begin{array}{ll} \text{minimize} & x_1 + 2x_2 + x_4 + x_5 - 5x_6 \\ \text{subject to} & \begin{array}{rcrcrcrcrcrcr} 6x_1 & - & 2x_2 & + & x_3 & - & x_4 & + & x_5 & + & 2x_6 & = & 4 \\ 2x_1 & - & \frac{1}{3}x_2 & - & x_3 & + & x_4 & + & \frac{1}{2}x_5 & & & = & 3 \\ -3x_1 & + & x_2 & - & 2x_3 & - & 4x_4 & - & \frac{1}{2}x_5 & - & x_6 & = & -2 \\ & & & & & & & & \multicolumn{3}{r}{x_1, x_2, x_3, x_4, x_5, x_6} & \geq & 0 \end{array} \end{array}$$

**Exercise 20.** **15 points**

Consider a standard form problem and assume that the rows of the constraint matrix $\mathbf{A}$ are linearly independent. For an $\varepsilon > 0$ define

$$\mathbf{b}(\varepsilon) := \mathbf{b} + \begin{pmatrix} \varepsilon \\ \varepsilon^2 \\ \vdots \\ \varepsilon^m \end{pmatrix}.$$

Then the $\varepsilon$-*perturbed problem* is obtained from the original by replacing $\mathbf{b}$ by $\mathbf{b}(\varepsilon)$.

a) Given a basis matrix $\mathbf{B}$, show that the corresponding basic solution $\mathbf{x}_B(\varepsilon)$ in the $\varepsilon$-perturbed problem is equal to

$$\mathbf{B}^{-1}\left(\mathbf{b} \mid \mathbf{I}\right) \begin{pmatrix} 1 \\ \varepsilon \\ \vdots \\ \varepsilon^m \end{pmatrix}.$$

b) Show that there exists some $\varepsilon^* > 0$ such that for $0 < \varepsilon < \varepsilon^*$ all basic solutions to the $\varepsilon$-perturbed problem are nondegenerate.

c) Suppose that all rows of $\mathbf{B}^{-1}\left(\mathbf{b} \mid \mathbf{I}\right)$ are lexicographically positive. Show that for $\varepsilon$ positive and sufficiently small, $\mathbf{x}_B(\varepsilon)$ is a basic feasible solution to the $\varepsilon$-perturbed problem.

d) Consider a feasible basis for the original problem and assume that all rows of $\mathbf{B}^{-1}\left(\mathbf{b} \mid \mathbf{I}\right)$ are lexicographically positive. Let some nonbasic variable $x_j$ enter the basis and define $\mathbf{u} = \mathbf{B}^{-1}\mathbf{A}_j$. Let the exiting variable be determined as follows: For every row $i$ such that $u_i$ is positive, divide the $i$-th row of $\mathbf{B}^{-1}\left(\mathbf{b} \mid \mathbf{I}\right)$ by $u_i$, compare the results lexicographically, and choose the exiting variable to be the one corresponding to the lexicographically smallest row. Show that this is the same choice of exiting variable as in the original simplex method applied to the $\varepsilon$-perturbed problem, provided $\varepsilon$ is sufficiently small.

e) Explain why the revised simplex method with the lexicographic rule described in part (d) is guaranteed to terminate even in the face of degeneracy.

**Exercise 21.** **(Tutorial session)**

To get a feasible start basis for the linear program $(*)$

$$\begin{array}{rl} \text{maximize} & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

($\mathbf{b}$ is not necessarily $\geq 0$), we consider the linear program $(**)$

$$\begin{array}{rrcl} \text{maximize} & z & & \\ \text{subject to} & \mathbf{A}\mathbf{x} - \mathbf{b}z & \leq & \mathbf{0} \\ & z & \leq & 1 \\ & \mathbf{x} & \geq & \mathbf{0} \\ & z & \geq & 0 \end{array}$$

different from the approach in the lecture. $(**)$ will be solved with the simplex algorithm. Describe a procedure that constructs a feasible basis for $(*)$ from an optimal basis of $(**)$ or decides that $(*)$ is unfeasible. Prove the correctness of your procedure and apply it to the linear programs

$$\begin{array}{rrcrcr} \text{maximize} & x_1 + x_2 & & & & \\ \text{subject to} & 2x_1 & - & x_2 & \leq & -1 \\ & x_1 & - & 2x_2 & \leq & -6 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

and

$$\begin{array}{rrcrcr} \text{maximize} & x_1 + x_2 & & & & \\ \text{subject to} & 2x_1 & - & x_2 & \leq & -1 \\ & -x_1 & + & 2x_2 & \leq & 0 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

**Exercise 22.** **(Tutorial session)**

While solving a standard form problem, we arrive at the following tableau, with $x_3$, $x_4$, and $x_5$ being the basic variables:

| | | | | | |
|---|---|---|---|---|---|
| $-10$ | $\delta$ | $-2$ | 0 | 0 | 0 |
| 4 | $-1$ | $\eta$ | 1 | 0 | 0 |
| 1 | $\alpha$ | $-4$ | 0 | 1 | 0 |
| $\beta$ | $\gamma$ | 3 | 0 | 0 | 1 |

The entries $\alpha$, $\beta$, $\gamma$, $\delta$, $\eta$ in the tableau are unknown parameters. For each of the following statements, find parameter values that will make the statement true.

a) The current solution is optimal and there are multiple optimal solutions.

b) The optimal cost is $-\infty$.

c) The current solution is feasible but not optimal.

**Exercise 23.** **(Tutorial session)**

Consider a linear programming problem in standard form, described in terms of the following initial tableau:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | $\delta$ | 3 | $\gamma$ | $\xi$ |
| $\beta$ | 0 | 1 | 0 | $\alpha$ | 1 | 0 | 3 |
| 2 | 0 | 0 | 1 | $-2$ | 2 | $\eta$ | $-1$ |
| 3 | 1 | 0 | 0 | 0 | $-1$ | 2 | 1 |

The entries $\alpha$, $\beta$, $\gamma$, $\delta$, $\eta$, $\xi$ in the tableau are unknown parameters. Furthermore, let $\mathbf{B}$ be the basis matrix corresponding to $x_2$, $x_3$, and $x_1$ (in that order) as basic variables. For each of the following statements, find the ranges of values of the various parameters such that the respective statement is true.

a) Phase II of the simplex method can be applied using this as an initial tableau.

b) The first row in the present tableau (below the reduced cost row) indicates that the problem is infeasible.

c) The corresponding basic solution is feasible, but we do not have an optimal basis.

d) The corresponding basic solution is feasible and the first simplex iteration indicates that the optimal cost is $-\infty$.

e) The corresponding basic solution is feasible, $x_6$ is a candidate for entering the basis, and when $x_6$ is the entering variable, $x_3$ leaves the basis.

f) The corresponding basic solution is feasible, $x_7$ is a candidate for entering the basis, but if it does, the solution and the objective value remain unchanged.