Technische Universität Berlin

Institut für Mathematik
Wintersemester 07/08

Linear and Integer Programming (ADM II)

Martin Skutella
Axel Werner

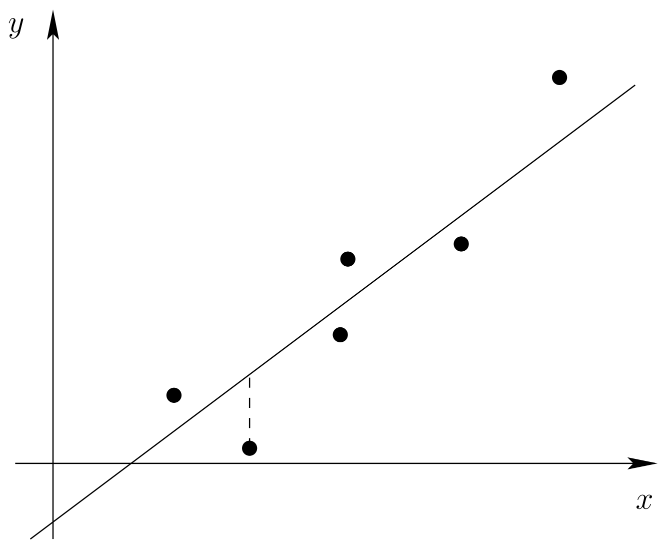
Torsten Ueckerdt
Jannik Matuschke

Exercise sheet 6

Deadline: Wed, 28 Nov 2007, 12:15 in MA 042

Exercise 24. **8 points**

Let $(x_1, y_1), \ldots, (x_n, y_n)$ be n points in a planar coordinate system. We are looking for a line ℓ with the equation $y = ax + b$, such that the largest vertical distance from ℓ to the points (x_i, y_i) is minimal.



a) Set up a linear program that models the problem.

b) Find the dual program to the one above.

c) Which one of the two programs would you solve if you had to solve it by hand and why?

please turn over

Exercise 25. **4 points**

Suppose we have a subroutine which, given a system of linear inequality constraints, either produces a solution or decides that no solution exists. Construct a simple algorithm that uses a single call to this subroutine to find an optimal solution to any linear programming problem that has an optimal solution.

(This shows that essentially solving linear programming problems is not harder than solving systems of linear inequalities.)

Exercise 26. **12 points**

a) Consider the following linear program:

$$\begin{array}{lrcrcrcl} \text{minimize} & x_1 + x_3 & & & & & & \\ \text{subject to} & x_1 & + & 2x_2 & & & \leq & 5 \\ & & & x_2 & + & 2x_3 & = & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Give an optimal solution to the linear program. Set up the dual linear program, state explicitly the conditions of complementary slackness and give an optimal solution to the dual program.

b) Consider the following linear program:

$$\begin{array}{lrcrcrcrcr} \text{minimize} & 2x_1 + 3x_2 + 4x_3 + 2x_4 & & & & & & & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & + & x_4 & \geq & 10 \\ & 3x_1 & + & x_2 & + & 4x_3 & + & 2x_4 & \geq & 12 \\ & & & & & & & x_1, x_2, x_3, x_4 & \geq & 0 \end{array}$$

Bring the linear program into standard form and argue why choosing the slack variables as basic variables yields a dual feasible solution. Solve the problem with the dual simplex method.

Exercise 27. **6 points**

Consider the dual simplex method applied to a standard form problem with linearly independent rows. Suppose that we have a basis which is primal infeasible, but dual feasible, and let i be such that $x_{B(i)} < 0$. Suppose that in the tableau all entries in the i-th row other than $x_{B(i)}$ are nonnegative.

Show that the optimal dual cost is $+\infty$.

Exercise 28. **(Tutorial session)**

What is wrong with the following argument:

By duality and the fact that $\min\{f(x) : x \in X\} \leq \max\{f(x) : x \in X\}$ we have

$$\begin{aligned}
\max\{\, \mathbf{c}^\top \mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \,\} &\leq \min\{\, \mathbf{p}^\top \mathbf{b} \mid \mathbf{p}^\top \mathbf{A} \geq \mathbf{c}^\top,\ \mathbf{p} \geq \mathbf{0} \,\} \\
&\leq \max\{\, \mathbf{p}^\top \mathbf{b} \mid \mathbf{p}^\top \mathbf{A} \geq \mathbf{c}^\top,\ \mathbf{p} \geq \mathbf{0} \,\} \\
&\leq \min\{\, \mathbf{c}^\top \mathbf{x} \mid \mathbf{A}\mathbf{x} \geq \mathbf{b},\ \mathbf{x} \leq \mathbf{0} \,\} \\
&\leq \max\{\, \mathbf{c}^\top \mathbf{x} \mid \mathbf{A}\mathbf{x} \geq \mathbf{b},\ \mathbf{x} \leq \mathbf{0} \,\} \\
&\leq \min\{\, \mathbf{p}^\top \mathbf{b} \mid \mathbf{p}^\top \mathbf{A} \leq \mathbf{c}^\top,\ \mathbf{p} \leq \mathbf{0} \,\} \\
&\leq \max\{\, \mathbf{p}^\top \mathbf{b} \mid \mathbf{p}^\top \mathbf{A} \leq \mathbf{c}^\top,\ \mathbf{p} \leq \mathbf{0} \,\} \\
&\leq \min\{\, \mathbf{c}^\top \mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \,\} \\
&\leq \max\{\, \mathbf{c}^\top \mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \geq \mathbf{0} \,\}.
\end{aligned}$$

Therefore every inequality holds with equality and there is no difference between minimizing and maximizing in a linear program.

Exercise 29. **(Tutorial session)**

Consider a linear programming problem in standard form which is infeasible, but which becomes feasible with a finite optimal cost when the last equality constraint is omitted. Show that the dual of the original (infeasible) problem is feasible and its optimal cost is infinite.

Exercise 30. **(Tutorial session)**

Give examples for linear programs that satisfy the following conditions.

a) The primal problem has a degenerate optimal basic feasible solution and the dual has a unique optimal solution.

b) Both the primal and the dual problems have more than one optimal solutions.

Exercise 31. **(Tutorial session)**

Decide with the conditions of complementary slackness whether $\mathbf{x}^* = (0, \frac{4}{3}, \frac{2}{3}, \frac{5}{3}, 0)^\top$ is an optimal solution of the following linear program.

$$\begin{array}{ll}
\text{maximize} & 7x_1 + 6x_2 + 5x_3 - 2x_4 + 3x_5 \\
\text{subject to} & \begin{array}{rcrcrcrcrcl}
x_1 & + & 3x_2 & + & 5x_3 & - & 2x_4 & + & 2x_5 & \leq & 4 \\
4x_1 & + & 2x_2 & - & 2x_3 & + & x_4 & + & x_5 & \leq & 3 \\
2x_1 & + & 4x_2 & + & 4x_3 & - & 2x_4 & + & 5x_5 & \leq & 5 \\
3x_1 & + & x_2 & + & 2x_3 & - & x_4 & - & 2x_5 & \leq & 1 \\
 & & & & \multicolumn{5}{r}{x_1, x_2, x_3, x_4, x_5} & \geq & 0
\end{array}
\end{array}$$