Technische Universität Berlin | Institut für Mathematik
Wintersemester 07/08

Linear and Integer Programming (ADM II)

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Exercise sheet 7

Deadline: Wed, 5 Dec 2007, 12:15 in MA 042

Exercise 32. **9 points**

A shipping company transports the delicious Yorkshire Dales Spring Water from a number of springs to various bottling factories in northern England. There are n springs which give rise to $a_1, \ldots, a_n$ pints of spring water per day. The water is then transported to one of m factories, which can bottle $b_1, \ldots, b_m$ pints every day. Transporting a pint of water from spring i to factory j costs c_{ij} Pounds.

The shipping company aims at managing all the necessary transports at the lowest possible cost.

a) Define and sketch a graph that models the constraints and set up a linear program for the problem.

b) State the dual linear program and give an interpretation for the objective function, the constraints and the shadow prices/marginal costs.

Exercise 33. **8 points**

Consider again the linear program from exercise 16:

$$\begin{array}{lrcrcl}
\text{minimize} & -2x_1 - x_2 & & & & \\
\text{subject to} & x_1 & - & 2x_2 & \leq & 2 \\
& x_1 & - & x_2 & \leq & 2 \\
& x_1 & + & x_2 & \leq & 6 \\
& -4x_1 & + & x_2 & \leq & 1 \\
& & & x_1, x_2 & \geq & 0
\end{array}$$

Suppose the right-hand side vector is changed to $(1, 3, 4, 1)^\top$ and the new constraint $2x_1 - x_2 \leq 2$ is added.

Show how the basis corresponding to your original optimal solution, which was (hopefully) the point $(x_1, x_2) = (4, 2)$, can be used to initialise the dual simplex method and solve the new linear program with this approach.

please turn over

Exercise 34. **12 points**

a) For each of the following polyhedra give the extreme points and extreme rays:

(i) $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1,\ x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0\}$

(ii) $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 - x_2 = 17,\ x_1 + x_2 = 42\}$

(iii) $\{(x_1, x_2) \in \mathbb{R}^2 \mid 4x_1 + 2x_2 \geq 8,\ 2x_1 + x_2 \leq 8\}$

b) Is it possible to express each of the points in the polyhedron in (iii) as a convex combination of its extreme points plus a nonnegative linear combination of its extreme rays? Is this compatible with the resolution theorem?

c) If P is a polyhedron with at least one extreme point, is it then possible to express an arbitrary point of P as a convex combination of its extreme points plus a nonnegative multiple of a single extreme ray?

Exercise 35. **5 points**

Consider a linear programming problem in standard form under the usual asumption that the rows of the matrix $\mathbf{A}$ are linearly independent. Suppose that the columns $\mathbf{A}_1, \ldots, \mathbf{A}_m$ form a basis for an optimal solution. Let $\overline{\mathbf{A}}$ be some vector of the correct dimension and suppose the column $\mathbf{A}_1$ is changed to $\overline{\mathbf{A}}_1 := \mathbf{A}_1 + \delta\overline{\mathbf{A}}$.

Consider the matrix $\mathbf{B}(\delta)$, consisting of the columns $\overline{\mathbf{A}}_1, \mathbf{A}_2, \ldots, \mathbf{A}_m$. Let $[\delta_1, \delta_2]$ be a closed interval such that $0 \in [\delta_1, \delta_2]$ and $\det \mathbf{B}(\delta) \neq 0$ for all $\delta \in [\delta_1, \delta_2]$.

Show that the subset of $[\delta_1, \delta_2]$ for which $\mathbf{B}(\delta)$ is an optimal basis is also a closed interval.

Exercise 36. **(Tutorial session)**

While solving a standard form linear program

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

using the simplex method, we arrive at the following tableau:

	x_1	x_2	x_3	x_4	x_5
	0	0	$\bar{c}_3$	0	$\bar{c}_5$
$x_2 = 1$	0	1	-1	0	β
$x_4 = 2$	0	0	2	1	γ
$x_1 = 3$	1	0	4	0	δ

Additionally, suppose that the last three columns of the matrix $\mathbf{A}$ form an identity matrix.

a) Give necessary and sufficient conditions for the coefficients in the tableau such that the basis described by this tableau is optimal.

b) Assume that this basis is optimal and that $\overline{c}_3 = 0$. Find an optimal basic feasible solution, other than the one described by this tableau.

c) Suppose that $\gamma > 0$. Show that there exists an optimal basic feasible solution, regardless of the values of $\overline{c}_3$ and $\overline{c}_5$.

d) Assume that the basis associated with the tableau is optimal. Suppose that b_1 in the original problem is replaced by $b_1 + \varepsilon$. Give upper and lower bounds on ε such that the basis remains optimal.

e) Assume that the basis associated with the tableau is optimal. Suppose that c_1 in the original problem is replaced by $c_1 + \varepsilon$. Give upper and lower bounds on ε such that the basis remains optimal.

Exercise 37. **(Tutorial session)**

Prove the following statements:

a) $\mathbf{Ax} < \mathbf{0}$ infeasible $\Longleftrightarrow$ there exists some $\mathbf{y} \geq \mathbf{0}$ with $\mathbf{y} \neq \mathbf{0}$ such that $\mathbf{y}^\top \mathbf{A} = \mathbf{0}$. (P. Gordan 1873)

b) $\mathbf{Ax} = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$ infeasible $\Longleftrightarrow$ there exists some $\mathbf{y}$ such that $\mathbf{y}^\top \mathbf{A} \geq \mathbf{0}$ and $\mathbf{y}^\top \mathbf{A} \neq \mathbf{0}$. (E. Stiemke 1915)

c) $\mathbf{Ax} < \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$ infeasible $\Longleftrightarrow$ there exists some $\mathbf{y} \geq \mathbf{0}$ with $\mathbf{y} \neq \mathbf{0}$ such that $\mathbf{y}^\top \mathbf{A} \geq \mathbf{0}$. (J. A. Ville 1938)

d) $\mathbf{Ax} \geq \mathbf{0}$, $\mathbf{x} \geq \mathbf{0}$ has no solution with $x_k > 0 \Longleftrightarrow$ there exists some $\mathbf{y} \geq \mathbf{0}$ such that $\mathbf{y}^\top \mathbf{A} \leq \mathbf{0}$ and $\sum_{i=1}^{m} a_{ik} y_i < 0$. (A. W. Tucker 1956)

The first

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will take place on **Thu, 6 Dec** at **18:00** in **Café Campus**, directly behind the Mathematics Building. Hope to see you there!