Linear and Integer Programming (ADM II)

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Exercise sheet 8

Deadline: Wed, 12 Dec 2007, 12:15 in MA 042

This Exercise sheet is the first of the second half of semester.

Exercise 38. **10 points**

Let $G = (V, E)$ be a directed graph and $s, t \in V$, $s \neq t$ two nodes. Additionally, let $c : E \to \mathbb{R}^+$ be a function that provides a nonnegative *capacity* $c(e)$ for every edge $e \in E$. A *flow* for the network (G, c) is a function $f : E \to \mathbb{R}^+$ such that

(i) $f(e) \leq c(e)$ for every edge $e \in E$ (the flow on e may not exceed the capacity of e) and

(ii) for all nodes $v \in V \setminus \{s, t\}$ we have

$$\sum_{(u,v)\in E} f(u, v) \;=\; \sum_{(v,w)\in E} f(v, w) \qquad \text{(flow conservation)},$$

where $(a, b) \in E$ is an edge with start node a and target node b.

The problem is to find a *maximal flow* on the network, that is a flow such that its *value*

$$\sum_{(s,u)\in E} f(s, u) \;-\; \sum_{(w,s)\in E} f(w, s)$$

is maximized.

a) Set up a linear program that models the problem and construct its dual.

b) For $S \subseteq V$ let $\delta^+(S) := \{(u, v) \in E : u \in S, v \notin S\}$ be the set of edges crossing between the set S and its complement in V. If $s \in S$ and $t \notin S$, then $\delta^+(S)$ is called an *s-t-cut* and the *capacity* of the cut is

$$c(S) \;:=\; \sum_{e\in\delta^+(S)} c(e).$$

Show that the value of a maximal flow for (G, c) is equal to the minimal capacity of an s-t-cut.
(**Hint:** You can assume that dual basic solutions are integral. This can also be shown, but requires a little effort...)

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Exercise 39. **6 points**

Consider again the linear program from exercise 16:

$$\begin{array}{lrcrcl} \text{minimize} & -2x_1 - x_2 & & & & \\ \text{subject to} & x_1 & - & 2x_2 & \leq & 2 \\ & x_1 & - & x_2 & \leq & 2 \\ & x_1 & + & x_2 & \leq & 6 \\ & -4x_1 & + & x_2 & \leq & 1 \\ & & & x_1, x_2 & \geq & 0 \end{array}$$

Calculate its optimum, together with the optimal solution (which was $(x_1, x_2) = (4, 2)$, as you might remember) by Fourier-Motzkin elimination.

Exercise 40. **10 points**

Consider the following linear program:

$$\begin{array}{lrcrcrcrl} \text{minimize} & 4x_1 & & & + & 5x_3 & & & \\ \text{subject to} & 2x_1 & + & x_2 & - & 5x_3 & & & = 1 \\ & -3x_1 & & & + & 4x_3 & + & x_4 & = 2 \\ & & & & & x_1, x_2, x_3, x_4 & & & \geq 0 \end{array}$$

a) Write down a simplex tableau and find an optimal solution. Is it unique?

b) Write down the dual problem and find a dual optimal solution. Is this unique?

c) Suppose that the right hand side vector $(1, 2)^\top$ is changed to $(1 - 2\theta, 2 - 3\theta)^\top$, where $\theta \in \mathbb{R}$ is a scalar parameter. Find an optimal solution and the value of the optimal cost, as a function of θ.

Exercise 41. **4 points**

For a given constraints matrix $\mathbf{A}$ define (as in section 5.2 of the lecture)

$$P(\mathbf{b}) := \{\mathbf{x} \mid \mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}\} \quad \text{and} \quad S := \{\mathbf{b} \mid P(\mathbf{b}) \neq \emptyset\}.$$

Furthermore, for any $\mathbf{b} \in S$, let

$$F(\mathbf{b}) := \min_{\mathbf{x} \in P(\mathbf{b})} \mathbf{c}^\top \mathbf{x}$$

be the optimal cost of the corresponding linear program as a function of $\mathbf{b}$. Show that for any $\lambda > 0$ and any $\mathbf{b} \in S$, we have $F(\lambda \mathbf{b}) = \lambda F(\mathbf{b})$. Assume that the dual feasible set is non-empty, so that $F(\mathbf{b})$ is finite.

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Exercise 42. **(Tutorial session)**

A paper factory manufactures large rolls of paper that have a width of 105cm. However, retailers demand rolls of smaller width, which have to be cut from the large ones. For instance, a standard width roll could be cut into two rolls of 35cm each and one roll of 30cm. The factory received the following orders:

width (cm)	no. of rolls
25	100
30	125
35	80

Naturally, the goal is to produce as few large rolls as possible while satisfying the demand.

a) Set up a linear program that models the problem.

b) Consider the general problem: The large rolls have width W and the demand is for b_i rolls of width $w_i \leq W$, where $i = 1, \ldots, m$. Explain how to obtain the constraints in the general case and convince yourself that the number of variables can get very large.

c) The full formulation that one gets out of the problem is an *integer program*—it requires as additional constraints that the variables have to be integer. Use an optimal solution for the original linear program to construct a feasible solution for the integer programming problem, whose cost differs from the optimal cost by at most m.