TECHNISCHE UNIVERSITÄT BERLIN INSTITUT FÜR MATHEMATIK
Wintersemester 07/08

Linear and Integer Programming (ADM II)

Martin Skutella
Axel Werner
Torsten Ueckerdt
Jannik Matuschke

Exercise sheet 9

Deadline: Wed, 19 Dec 2007, 12:15 in MA 042

Exercise 43. **12 points**

The small Berlin company **VANPEY** produces hand-made furniture. The company has its head office in Prenzlberg, while production of paper holders and stools is done in a Kreuzberg branch and small tables and stylish bookshelfs are produced in a workshop in Friedrichshain. The two branches are again divided into two teams each (team **A** and **B** for the Kreuzberg branch and **C** and **D** for the one in Friedrichshain).

The products require certain amounts of raw material and working hours:

- Each paper holder requires 1 unit of wood, 1 unit of metal, 2 hours in team **A**, 1 hour in team **B** and brings in a net profit of €30.
- Each stool requires 2 units of wood, 1 unit of metal, 3 hours in team **A**, 1 hour in team **B** and brings in a net profit of €40.
- 120 hours in team **A** and 50 hours in team **B** are available per day.
- Each table requires 2 units of wood, 1 unit of metal, 1 hour in team **C**, 2 hours in team **D** and brings in a net profit of €40.
- Each bookshelf requires 3 units of wood, 1 unit of metal, 3 hours in team **C**, 1 hour in team **D** and brings in a net profit of €70.
- 150 hours in team **C** and 100 hours in team **D** are available per day.

The operations of the two workshops are not entirely independent: only 225 units of wood and 114 units of metal per day are available to the entire company. Under the assumption that all the produced furniture can be sold, the company wants to maximise its profit.

a) Set up a linear program for the problem and write it in a way such that Dantzig-Wolfe decomposition can be applied.

b) Solve the linear program with the Dantzig-Wolfe decomposition algorithm and interpret each iteration in economic terms. (You can solve the subproblems with a computer.)

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Exercise 44. **6 points**

Consider the Dantzig-Wolfe decomposition method and suppose that we are at a basic feasible solution to the master problem.

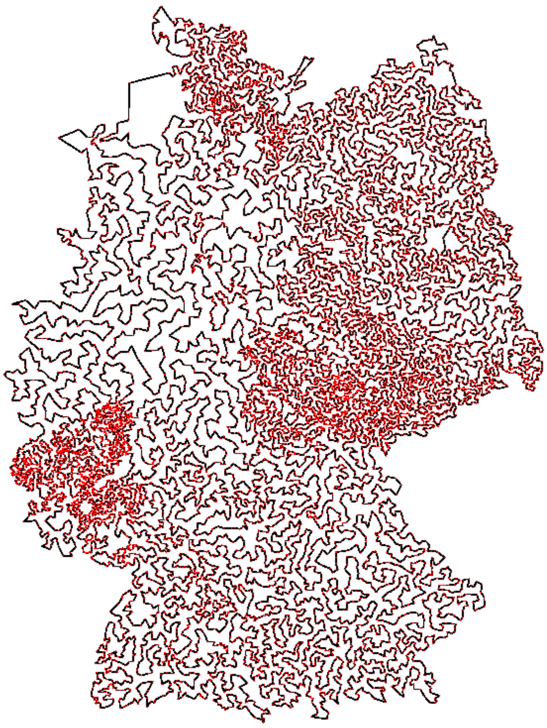
a) Show that at least one of the variables λ_1^j must be a basic variable.

b) Let r_1 be the current value of the simplex multiplier associated with the first convexity constraint $\sum_{j\in J_1} \lambda_1^j = 1$, and let z_1 be the optimal cost in the first subproblem. Show that $z_1 \leq r_1$.

Exercise 45. **10 points**

The *Traveling Salesperson Problem (TSP)* can be treated (though not completely solved) as a linear programming problem: Consider n cities that have to be visited by a salesperson. The problem is to find a shortest round-trip that passes through each city exactly once. See the website `http://www.tsp.gatech.edu/index.html` for lots of information about TSP, its history, pictures and games!

In a mathematical formulation the problem reads as follows: Given a complete graph $K_n = (V, E)$ on n nodes, together with costs (distances) $c_e \geq 0$ for every edge $e \in E$, one wants to find a (Hamiltonian) cycle $\mathcal{C}$ of minimum total cost $\sum_{e\in\mathcal{C}} c_e$ such that all nodes of K_n are visited.



Consider the following linear program (LP):

$$\begin{array}{lrcll} \text{minimize} & \displaystyle\sum_{e\in E} c_e x_e & & & \\ \text{subject to} & \displaystyle\sum_{e\in\delta(v)} x_e & = & 2 & \forall v \in V \\ & \displaystyle\sum_{e\in\delta(S)} x_e & \geq & 2 & \forall \emptyset \neq S \subsetneq V \\ & x_e & \geq & 0 & \forall e \in E \end{array}$$

Here $\delta(S) = \{\{i,j\} \mid i \in S, j \notin S\}$ is the set of edges that leave the vertex set S; in particular, $\delta(v)$ is the set of edges incident to a given vertex v.

a) Show that every solution of TSP is also a solution of LP with the same cost, and explain the constraints in the linear program. Note that there are $2^n - 2$ inequality constraints coming from all nontrivial subsets of V. Why are these constraints needed?

b) The linear program can be solved by a cutting plane method. Suppose we are given a vector $\mathbf{x} \geq \mathbf{0}$ that satisfies all equality constraints. The *separation problem* is then to either find a set S such that the corresponding inequality constraint is violated or verify that all inequality constraints hold. Show that this problem can be solved without explicitly enumerating all the (exponentially many) inequalities.
(*Hint:* Exercise 38.)

Exercise 46. **(Tutorial session)**

Consider a linear programming problem of the form

$$\begin{array}{ll} \text{minimize} & \mathbf{c}_1^\top \mathbf{x}_1 + \mathbf{c}_2^\top \mathbf{x}_2 + \mathbf{c}_0^\top \mathbf{y} \\ \text{subject to} & \begin{pmatrix} \mathbf{D}_1 & \mathbf{0} & \mathbf{F}_1 \\ \mathbf{0} & \mathbf{D}_2 & \mathbf{F}_2 \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{y} \end{pmatrix} \geq \begin{pmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{pmatrix} \\ & \mathbf{x}_1, \mathbf{x}_2 \ \geq \ \mathbf{0} \end{array}$$

There are two different ways of decomposing this problem.

a) Form the dual problem and explain how Dantzig-Wolfe decomposition can be applied to it. What is the structure of the subproblems solved during a typical iteration?

b) Rewrite the first set of constraints in the form $\mathbf{D}_1\mathbf{x}_1 + \mathbf{F}_1\mathbf{y}_1 \geq \mathbf{b}_1$ and $\mathbf{D}_2\mathbf{x}_2 + \mathbf{F}_2\mathbf{y}_2 \geq \mathbf{b}_2$, together with a constraint relating $\mathbf{y}_1$ to $\mathbf{y}_2$. Discuss how to apply Dantzig-Wolfe decomposition and describe the structure of the subproblems solved during a typical iteration.

Exercise 47. **(Tutorial session)**

Suppose we are given n foods and m nutrients and an $m \times n$ matrix $\mathbf{A} = (a_{ij})_{i,j}$, where $a_{ij} \geq 0$ specifies the amount of nutrient i that is contained in one unit of the j-th food.

Consider a parent with two children. Let $\mathbf{b}$ and $\mathbf{b}'$ be the minimal nutritional requirements of the two children, respectively, and let $\mathbf{c}$ be the cost vector with the prices of the different foods. Assume that $c_i > 0$ for all i.

The parent has to buy food to satisfy the children's needs, at minimum cost. To avoid jealousy, there is the additional constraint that the amount to be spent for each child is the same.

a) Provide a standard form formulation of this problem. What are the dimensions of the constraint matrix?

b) Suppose the Dantzig-Wolfe decomposition method is used to solve the problem in part a). Construct the subproblems solved during a typical iteration of the master problem.

c) Suggest a direct approach for solving this problem based on the solution of two single-child diet problems.