Technische Universität Berlin

Institut für Mathematik
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Linear and Integer Programming (ADM II)

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Exercise sheet 10

Deadline: Wed, 9 Jan 2008, 12:15 in MA 042

Exercise 48. **10 () points**

Santa Claus's cookie kitchen needs to prepare the cookies for this year's christmas. The recipe book of Santa's grandma Mary contains recipes for n different cookie types, ranging from simple chocolate cookies to intricate almond and peanut butter cookies, even raisin and rum cookies (for the older children).

Santa needs at least c_i cookies of type i that will then be packed by his gift-wrapping elves and distributed to the well-behaved children by himself. There are m ingredients and each cookie of type i needs an amount of a_{ki} units of ingredient k. Thanks to Santa's shopping elves all the ingredients are already in stock, namely z_k units of each ingredient k. It is important that everything is used up to make room for Easter Bunny's easter eggs after christmas.

Baking one cookie of type i takes t_i minutes and we assume that the baking elves are not very bright, so they need $\ell \cdot t_i$ minutes to bake ℓ cookies of type i. Unfortunately, christmas is drawing near, so Santa wants to get everything done as soon as possible.

Help Santa's planning elves by setting up a linear program that solves the problem.

Exercise 49. **8 points**

A polyhedron $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{Ax} \geq \mathbf{b}\}$ is defined to be *full-dimensional* if it has positive volume. Assume that P is bounded and all rows of $\mathbf{A}$ are nonzero. Show that the following statements are equivalent:

(i) P is full-dimensional.

(ii) There exists a point $\mathbf{x} \in P$ such that $\mathbf{Ax} > \mathbf{b}$.

(iii) There are $n+1$ extreme points of P that do not lie on a common hyperplane.

please turn over

Exercise 50. **5 points**

How can one reduce problems of the form

$$\text{Find } \mathbf{x} \in \mathbb{R}^n \text{ s.t. } A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \leq \mathbf{0}, \quad A \in \mathbb{Q}^{m \times n},\ \mathbf{b} \in \mathbb{Q}^m$$

to problems of the form

$$\text{Find } \mathbf{x} \in \mathbb{R}^n \text{ s.t. } A\mathbf{x} \leq \mathbf{b},\ \mathbf{x} \leq \mathbf{0}, \quad A \in \mathbb{Z}^{m \times n},\ \mathbf{b} \in \mathbb{Z}^m?$$

How does the input length in the bit model change with this reduction?

Exercise 51. **7 points**

Let $s \in \mathbb{N}$ and

$$\mathbf{A} := \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 2^s & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{b} := \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Furthermore, let $P := \{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$. Find a feasible solution with the ellipsoid method for $s = 0$ and $s = 1$.

Exercise 52. **(Tutorial session)**

Let $P = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \geq \mathbf{b}\}$ be a bounded full-dimensional polyhedron with at least one extreme point. Consider the linear program

$$\begin{aligned} \text{minimize} \quad & \mathbf{c}^\top \mathbf{x} \\ \text{subject to} \quad & \mathbf{A}\mathbf{x} \geq \mathbf{b} \end{aligned}$$

and let z be the optimal cost. We assume that the entries of $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ are integer, and have absolute value bounded by U. Let $\varepsilon > 0$. Prove that the sliding objective ellipsoid method finds a solution $\mathbf{x}_t$ with cost $\mathbf{c}^\top \mathbf{x}_t \leq z + \varepsilon$ after a number of iterations t polynomial in n, $\log U$ and $\log(1/\varepsilon)$.

Exercise 53. **(Tutorial session)**

Given an undirected graph $G = (V, E)$, two distinct nodes $s, t \in V$ and weights c_e for all $e \in E$, let $\mathcal{K}$ be the set of all paths from s to t. We would like to find a minimum weight set of edges that intersects every path in $\mathcal{K}$. One way of formulating this problem is as follows:

$$\begin{aligned} \text{minimize} \quad & \sum_{e \in E} c_e x_e \\ \text{subject to} \quad & \sum_{e \in K} x_e \geq 1 \quad \forall\, K \in \mathcal{K} \\ & 0 \leq x_e \leq 1 \quad \forall\, e \in E \end{aligned}$$

One can show that the extreme points of the feasible set are vectors in $\{0, 1\}^E$, and that an optimal basic feasible solution indeed corresponds to a minimum cut. Prove that the associated separation problem can be solved in polynomial time.

Exercise 54. **(Tutorial session)**

Given a complete undirected graph $G = (V, E)$ with weights c_e for $e \in E$ and a function $f : 2^V \to \mathbb{R}$, consider the problem

$$\begin{array}{lll} \text{minimize} & \sum_{e \in E} c_e x_e & \\ \text{subject to} & \sum_{e \in \delta(S)} x_e \geq f(S) & \forall\, \emptyset \neq S \subsetneq V \\ & 0 \leq x_e \leq 1 & \forall\, e \in E \end{array}$$

The above linear programming problem provides a lower bound for the optimal cost of the corresponding *integer* programming problem, in which x_e must take values 0 or 1. Solve the separation problem for the linear programs corresponding to the following cases.

a) Let
$$f(S) := \max_{e \in \delta(S)} r_e$$
where $r_e \geq 0$ for all $e \in E$. The corresponding integer programming problem is called the *survivable network design* problem.

b) Given a set $T \subset V$, let
$$f(S) := \begin{cases} 1, & \text{if } S \cap T \neq \emptyset, T, \\ 0, & \text{otherwise.} \end{cases}$$
The corresponding integer programming problem is called the *Steiner tree* problem.

Exercise 55. **(Tutorial session)**

Find a 2-dimensional ellipsoid of minimal volume that contains the positive semi-disc $\{\mathbf{x} \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \leq 1, x_2 \geq 0\}$.

WE WISH EVERYONE A WONDERFUL CHRISTMAS AND A GREAT YEAR 2008!