Linear and Integer Programming (ADM II)

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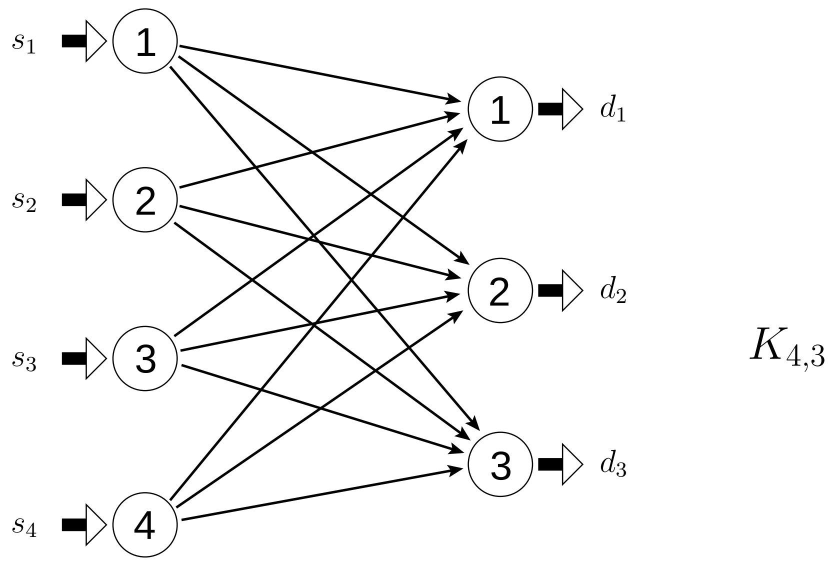
Exercise sheet 11

Deadline: Wed, 16 Jan 2008, 12:15 in MA 042

Exercise 56. **8 points**

A *transportation problem* is a special network flow problem, where the network is a *bipartite graph* $K_{N,M}$ with N *suppliers* and M *consumers*. For every supplier i, $1 \le i \le N$ we have a supply of $s_i \ge 0$ and analogously a demand $d_j \le 0$ for every consumer j, $1 \le j \le M$. Additionally there is an edge of non-negative cost from every supplier to every consumer. See exercise 32. for an example!



Consider now an uncapacitated network flow problem and assume that $c_{(i,j)} \ge 0$ for all arcs (i, j). Let S_+ and S_- be the sets of source and sink nodes, respectively. For $i \in S_+$ and $j \in S_-$ let ν_{ij} be the cost of a shortest directed path from node i to node j or ∞ if no path exists.

Construct from this general network flow problem a transportation problem with the same source and sink nodes, and the same values for the supplies and demands by introducing an arc (i, j) for every $i \in S_+$ and $j \in S_-$ with cost ν_{ij}. Show that the two problems have the same optimal cost.

please turn over

Exercise 57. **8 points**

Suppose that we are given a non-integer optimal solution to an uncapacitated network flow problem with integer data.

(a) Show that there exists a cycle with every arc on the cycle carrying a positive flow. What can you say about the cost of such a cycle?

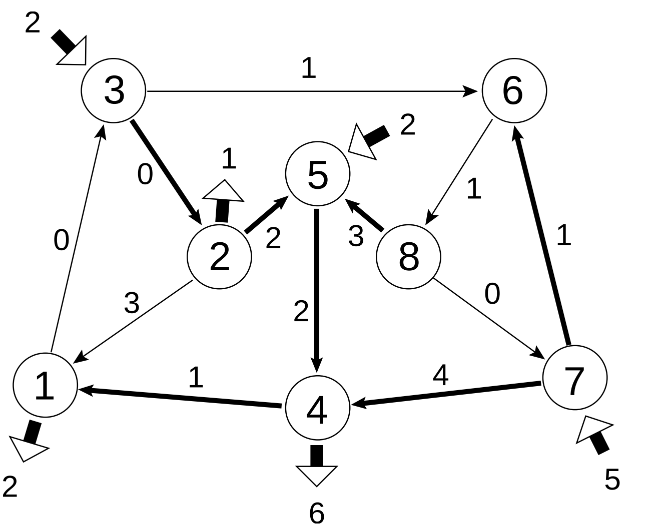
(b) Suggest a method for constructing an integer optimal solution, without solving the problem from scratch.
(*Hint:* Remove cycles.)

Exercise 58. **6 points**

Consider a network flow problem in which we impose an additional lower bound constraint $f_{(i,j)} \geq d_{(i,j)} \geq 0$ for every arc (i, j). Construct an equivalent network flow problem in which there are no nonzero lower bounds on arc flows.
(*Hint:* Let $\overline{f}_{(i,j)} = f_{(i,j)} - d_{(i,j)}$ and construct a new network for the arc flows $\overline{f}_{(i,j)}$. How should b_i be changed?)

Exercise 59. **8 points**

Consider the uncapacitated network flow problem below, where the costs are given for each arc.



(a) Give the incidence matrix **A** corresponding to the problem.

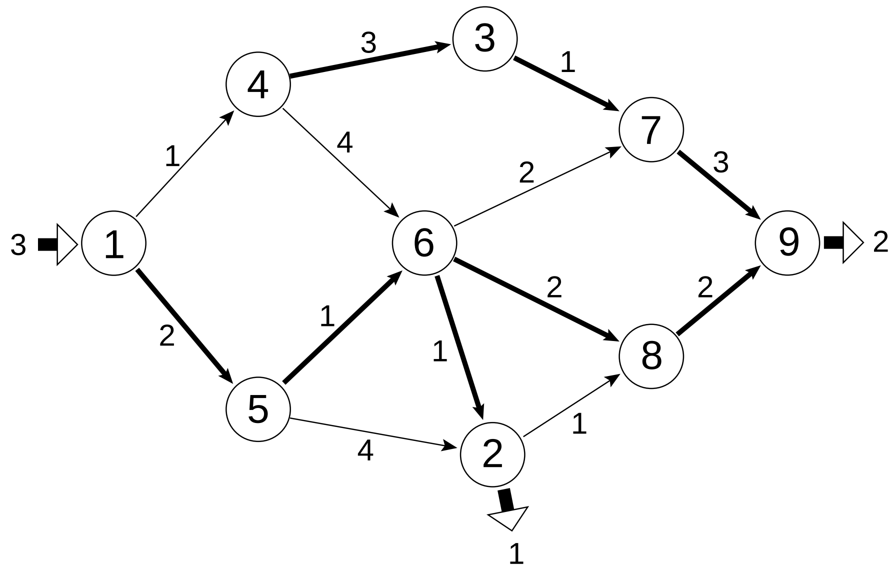
(b) Solve the problem using the network simplex algorithm. Start with the tree indicated by the thicker edges.

Exercise 60. **(Tutorial session)**

Consider a transportation problem in which all cost coefficients $c_{(i,j)}$ are positive. Suppose that we increase the supply at some source nodes and the demand at some sinks. (In order to maintain feasibility, we assume that the increases are such that total demand is equal to total supply.) Is it true that the value of the optimal cost will also increase? Prove or provide a counterexample.

Exercise 61. **(Tutorial session)**

For the uncapacitated network flow problem below, where the costs are given for each arc, consider the spanning tree indicated by the thicker edges and the associated tree solution.



(a) What are the values of the arc flows corresponding to this basic solution? Is it a feasible basic solution?

(b) For this basic solution, find the reduced cost of each arc in the network.

(c) Is this basic solution optimal?

(d) Does there exist a non-degenerate optimal basic feasible solution?

(e) Find an optimal dual solution.

(f) By how much can we increase the cost of arc $(5, 6)$ and still have the same optimal basic feasible solution.

(g) If we increase the supply at node 1 and the demand at node 9 by a small positive amount δ, what is the change in the value of the optimal cost?

(h) Does this problem have a special structure that makes it simpler than the general uncapacitated network flow problem?