Technische Universität Berlin

Institut für Mathematik
Wintersemester 07/08

Linear and Integer Programming (ADM II)

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Exercise sheet 13

Deadline: Wed, 30 Jan 2008, 12:15 in MA 042

Exercise 66. **12 points**

Suppose we have n jobs that can be carried out by a machine. Job j takes p_j units of time, where p_j is integer, and we want to find a *single-machine schedule*, that is an order in which the jobs are put onto one machine, that minimizes the *average completion time* $\frac{1}{n}\sum_{j=1}^{n} c_j$. Here the *completion time of job* j is $c_j := \min\{t \mid \text{job } j \text{ is finished at time } t\}$, where the first scheduled job starts at time 0. Additionally, we assume that jobs may not be interrupted once they are started, and the machine can only carry out one job at every time.

(a) Show that the problem can be formulated as a network flow problem.

(b) Give another linear program that also models the problem by considering the variables

$$x_{j,t} := \begin{cases} 1 & \text{if job } j \text{ starts at time } t \\ 0 & \text{otherwise} \end{cases}$$

(One constraint will be that $x_{j,t} \in \{0,1\}$, so this is in fact an integer program...)

(c) Find an optimal solution by guessing and prove that it is optimal.

Exercise 67. **8 points**

Consider the linear program

$$\begin{array}{lrcrcrcrcl} \text{minimize} & \multicolumn{9}{l}{6x_1 + 8x_2 + 5x_3 + 9x_4} \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & + & x_4 & = & 4 \\ & & & & & \multicolumn{3}{r}{x_1, x_2, x_3, x_4} & \geq & 0 \end{array}$$

(a) Perform the first step of the primal path following method, starting with the primal feasible solution $\mathbf{x} = (1,1,1,1)^\top$, the dual feasible solution $p = 0$ and the parameters $\mu^0 = 4$ and $\alpha = 3/4$. Compare the duality gaps before and afterwards. What is the optimal solution to the linear program?

(b) What happens if you start with $\mu^0 = 2$ (alternatively, if you perform the second step from part (a)). Give an explanation and a possibility to prevent this.

Exercise 68. **6 points**

Consider the linear program

$$\begin{array}{rrcrcrcl} \text{minimize} & x_1 & + & x_2 & & & & \\ \text{subject to} & x_1 & + & x_2 & + & x_3 & = & 1 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

(a) Show that the central path is given by

$$\begin{aligned} x_1(\mu) &= \frac{1 + 3\mu - \sqrt{9\mu^2 - 2\mu + 1}}{4} \\ x_2(\mu) &= x_1(\mu) \\ x_3(\mu) &= 1 - 2x_1(\mu) \end{aligned}$$

(b) Show that for $\mu \to 0$ the points on the central path converge to the unique optimal solution of the linear program.

Exercise 69. **(Tutorial session)**

Consider the primal barrier problem with $\mu > 0$:

$$\begin{array}{ll} \text{minimize} & \mathbf{c}^\top \mathbf{x} - \mu \sum_{j=1}^{n} \log x_j \\ \text{subject to} & \mathbf{A}\mathbf{x} = \mathbf{b} \end{array}$$

(a) Show that the barrier function is strictly convex.

(b) Suppose the barrier problem has an optimal solution $\mathbf{x}(\mu)$. Show that it is the unique optimal solution. What can be said if there is no optimal solution?

(c) Show that if there exists a $\mu > 0$ such that the barrier problem has an optimal solution then it has an optimal solution for *all* $\mu > 0$.