Technische Universität Berlin

Institut für Mathematik
Wintersemester 07/08

Linear and Integer Programming (ADM II)

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Exercise sheet 14

Deadline: Wed, 6 Feb 2008, 12:15 in MA 042

Exercise 70. **5 points**

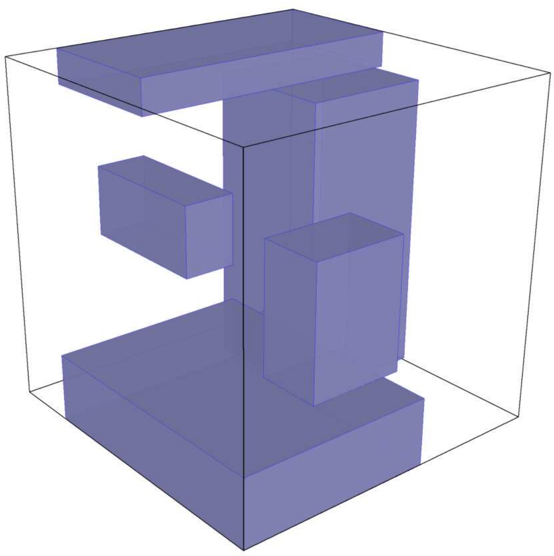
Why Santa Claus can afford to bring more presents than the Easter Bunny.

It's the time of year where Santa Claus and the Easter Bunny don't have anything to do, so they are killing time until their next public appearance by playing Rock, Paper, Scissors. The Easter Bunny, however, with his paw cannot indicate a pair of scissors and is therefore limited to only two pure strategies.

Set up the payoff matrix and compute the Nash equilibrium and game value for their game by linear programming.

Exercise 71. **10 points**

A number of bricks of different sizes have to be packed into a large box, in such a way that the sum of all side lengths of the box is minimized. Each brick has to be positioned with its edges parallel to the edges of the whole box and naturally different bricks may not overlap.



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(a) Model the problem as an integer program.

(b) Try to solve the problem (using `ZIMPL` and `CPLEX`) for five cubes of side lengths $1, \ldots, 5$.

Exercise 72. **5 points**

Let $P := \{(x, y)^\top \in \mathbb{R}^2 \mid y \leq \sqrt{2}x\}$. Prove that P_I is not a polyhedron.

Exercise 73. **10 points**

(*Facility location problem.*) Suppose we are given n potential facility locations and a list of m clients who need to be serviced from these locations. There is a fixed cost c_j of opening a facility at location j, while there is a cost of d_{ij} of serving client i from facility j. The goal is to select a set of facility locations and assign each client to one facility, while minimizing the total cost.

To model this problem we consider the following integer program $(*)$:

$$\begin{array}{rll}
\text{minimize} & \displaystyle\sum_{j=1}^{n} c_j y_j + \sum_{i=1}^{m}\sum_{j=1}^{n} d_{ij} x_{ij} & \\
\text{subject to} & \displaystyle\sum_{j=1}^{n} x_{ij} = 1 & \forall i \\
& x_{ij} \leq y_j & \forall i, j \\
& x_{ij}, y_j \in \{0, 1\} & \forall i, j
\end{array}$$

(a) Explain the variables and constraints in the above integer program and show that the optimal solution solves the facility location problem.

(b) Consider the modified integer program $(**)$, where the ($m \cdot n$ many) constraints $x_{ij} \leq y_j$ are replaced by the (n) constraints $\sum_{i=1}^{m} x_{ij} \leq m \cdot y_j$, $j = 1, \ldots, n$. Show that the sets of feasible solutions of the two formulations coincide.

(c) Let $P^{(*)}$ and $P^{(**)}$ be the feasible sets of the linear programming relaxations of the integer programs $(*)$ and $(**)$, respectively. Show that $P^{(*)} \subsetneqq P^{(**)}$.

Exercise 74. **(Tutorial session)**

Show the following statements:

(a) Let $X \subseteq \mathbb{R}^n$ and $\mathbf{y}$ some point in the convex hull $\mathsf{conv}\, X$. Then there are $\mathbf{x}_1, \ldots, \mathbf{x}_{n+1} \in X$ such that $\mathbf{y} \in \mathsf{conv}\{\mathbf{x}_1, \ldots, \mathbf{x}_{n+1}\}$. (**Carathéodory**, 1911)

(b) Let $X \subseteq \mathbb{R}^n$ and $\mathbf{y}$ a point in the polyhedral cone

$$C := \mathsf{cone}\, X := \left\{ \sum_{j=1}^{m} \lambda_j \mathbf{x}_j \mid m \geq 1, \lambda_1, \ldots, \lambda_m \geq 0, \mathbf{x}_1, \ldots, \mathbf{x}_m \in X \right\}.$$

Then there are $\mathbf{x}_1, \ldots, \mathbf{x}_n \in X$ such that $\mathbf{y} \in \mathsf{cone}\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$.

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(c) Recall that every polyhedral cone C as above can also be written as $C = \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{0}\}$ with some $m \times n$ matrix $\mathbf{A}$. Let $H = \{\mathbf{a}_1, \ldots, \mathbf{a}_t\}$ be a Hilbert basis of C. Then for every integral point $\mathbf{y} \in C$ there are $2n - 1$ vectors in H such that $\mathbf{y}$ is a nonnegative integer combination of those.

Exercise 75. **(Tutorial session)**

Consider the following two-player game: Each player independently writes down a number between 1 and 6 on a sheet of paper. Then the numbers are revealed and compared. If both numbers are equal, the game is a draw. If they differ by one, the player with the smaller number gets €2 from the other, and if the numbers differ by at least two, the player with the larger number gets €1.

Compute the optimal mixed strategies for the players in the game.

Exercise 76. **(Tutorial session)**

Suppose we are given m constraints $\mathbf{a}_i^\top \mathbf{x} \geq b_i$, with $\mathbf{a}_i \geq \mathbf{0}$ for all $i = 1, \ldots, m$. Model the requirement that at least k of the m constraints are satisfied.

How can this be solved without the nonegativity condition for the $\mathbf{a}_i$, but instead you can assume that there is a number f such that $\mathbf{a}_i^\top \mathbf{x} \geq f$ for all $i = 1, \ldots, m$ and all feasible $\mathbf{x}$.

Announcements

The second *Umtrunk* is scheduled for **Wed, 6 Feb 2008, 18:00** at Café Campus.

The **Dies Mathematicus** will take place on **Thu, 7 Feb 2008**, starting **14:00** (replacing the lecture), featuring interesting *talks on diploma theses*, as well as tales about *mathematics in the real world*.

`http://www.math.tu-berlin.de/dies/2007/`

Starting at 19:00: Party and live music by *The Roovers*!

`http://www.roovers.de/`