TECHNISCHE UNIVERSITÄT BERLIN

INSTITUT FÜR MATHEMATIK
Wintersemester 07/08

Linear and Integer Programming (ADM II)

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Exercise sheet 15

Deadline: **Thu**, 14 Feb 2008, **14:15** in MA 041

All points from this exercise sheet count as bonus points.
Therefore 100% of all points of semester equals 407 points, while 100% of all points from second half equals 196 points.

Exercise 77. **4 points**

We are given a set $V = \{1, \ldots, n\}$ of nodes (of a directed graph to-be) and demands or supplies, respectively, b_i for each $i \in V$, such that $\sum_{i=1}^{n} b_i = 0$. Furthermore, there are two types of costs: building costs d_{ij} for establishing a (directed) link (i, j) of capacity u_{ij} and transportation costs c_{ij} for shipping one unit from node i to node j, provided the arc (i, j) exists.

We would like to build a network in order to minimize the total building and transportation costs, so that all demands are met. Formulate this problem as an integer programming problem.

Exercise 78. **6 points**

Let the undirected graph $G = (V, E)$ with $V = \{1, \ldots, n\}$ represent a transportation network, where node $i = 1$ represents a central depot and node $i \in V$ $(i \neq 1)$ represents a customer with a demand of b_i units. A company has m vehicles, each of capacity Q, that need to visit all customers in order to satisfy demand. Each vehicle is to follow a route that starts at the central depot, visits some customers and afterwards returns to the depot. The travel costs are given by d_e for every arc $e \in E$.

Suppose that the demand of each customer can be carried by a single vehicle, i.e. $b_i \leq Q$ for all i. Assuming that the demand of any customer cannot be divided into more than one vehicle, formulate the problem of constructing routes for the vehicles that minimize the total transportation costs.

Exercise 79. **3 points**

For a given feasible and bounded integer program that minimizes the objective function let Z_{IP} denote the optimal value of the integer program and Z_{LP} the optimal value of its linear programming relaxation.

Obviously, $Z_{LP} \leq Z_{IP}$ for every integer program. Does there exist an $a > 0$ such that $Z_{IP} \leq a Z_{LP}$ for every integer program?

Exercise 80. **10 points**

Consider the integer programming problem

$$\begin{array}{lrcrcl}
\text{maximize} & x_1 + 2x_2 & & & & \\
\text{subject to} & -3x_1 & + & 4x_2 & \leq & 4 \\
& 3x_1 & + & 2x_2 & \leq & 11 \\
& 2x_1 & - & x_2 & \leq & 5 \\
& & & x_1, x_2 & \geq 0 & \\
& & & x_1, x_2 & \text{integer} &
\end{array}$$

Sketch the feasible set and use the figure to answer the following questions:

(a) What is the optimal cost of the linear programming relaxation? What is the optimal cost of the integer programming problem and how large is the gap between the two values?

(b) What is the convex hull of the feasible set of the integer program.

(c) Calculate and illustrate the first Gomory cut in this example. What is the optimal solution of the linear program after this cut?

Exercise 81. **7 points**

A matrix $\mathbf{A}$ is called *totally unimodular*, if $\det \mathbf{B} \in \{-1, 0, 1\}$ for every square submatrix $\mathbf{B}$ of $\mathbf{A}$.

Show that for a totally unimodular matrix $\mathbf{A}$ and integer vectors $\mathbf{b}$ and $\mathbf{u}$ the polyhedra

$$P := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}\} \quad \text{and} \quad P' := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{A}\mathbf{x} \leq \mathbf{b},\ \mathbf{0} \leq \mathbf{x} \leq \mathbf{u}\}$$

have only integer vertices.

Announcements

Schedule for the last week of semester:

- Monday, MA 041, 12-14: (last) exercise session
- Thursday, MA 041: 14:15–15:00: (last) lecture
  15:00: first meeting for next semester's seminar
- Program presentations can be done on Tue, Wed, Thu, 16-18, and additionally on Wed, 12-14 — please sign up on the list on Monday before exercise session.
- There will be *no lecture* on Wednesday and *no tutorial sessions* on Thursday and Friday.

⋆ The End ⋆