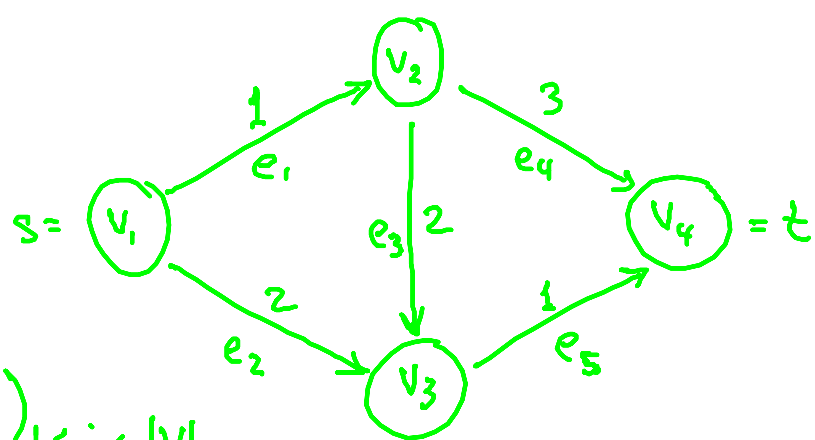
Umtrunk

~~Wed, 5.12. 18:00~~
Thu, 6.12. 18:00 Café Campus

Duality

An Example from combinatorial optimization.

- Graph $G=(V,E)$ connected, directed edges, weights $c(e)$ $\forall e \in E$ (cost)
  pick two vertices $s, t \in V$
  problem: find a path W from s to t with minimal cost $\sum_{e \in W} c(e)$



- Incidence Matrix $A = (a_{ij})_{\substack{1 \le i \le |V| \\ 1 \le j \le |E|}}$

$$a_{ij} := \begin{cases} -1 & \text{if } \xrightarrow{e_j} v_i \\ +1 & \text{if } v_i \xrightarrow{e_j} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & -1 \end{pmatrix}$$

$$\longrightarrow \sum_{v_i \in V} a_{ij} = 0 \quad \forall j$$

in fact: rank $A = |V| - 1$

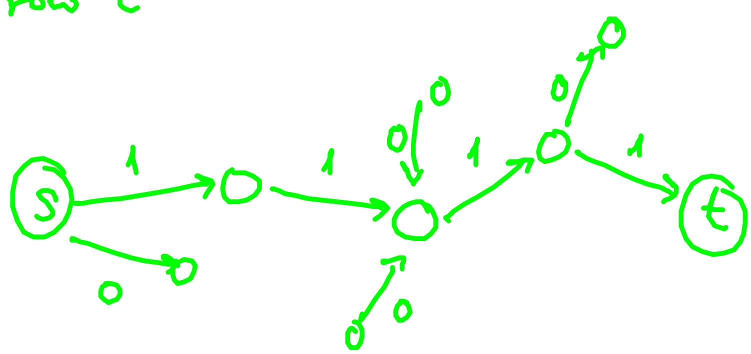
▸ $f \in \mathbb{R}^{|E|}$, $f \geq 0$ $\longrightarrow$ interpretation: f_j flow through edge e_j

▸ flow conservation: for all vertices v_i (except s, t)

$$\sum_{\substack{e_j \in E \\ \xrightarrow{e_j} v_i}} f_j = \sum_{\substack{e_j \in E \\ v_i \xrightarrow{e_j}}} f_j \iff \sum_{j=1}^{|E|} a_{ij} \cdot f_j = 0$$

$$\rightsquigarrow \quad A \cdot f = \begin{array}{cl} \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \alpha \\ 0 \\ \vdots \\ 0 \\ -\beta \\ 0 \\ \vdots \\ 0 \end{array}\right) & \begin{array}{l} \\ \\ \\ \leftarrow \text{row } s \\ \\ \\ \\ \leftarrow \text{row } t \\ \\ \\ \\ \end{array} \end{array}$$

▸ path from s to t is a flow $f \in \{0, 1\}^{|E|}$ with



$$A \cdot f = \begin{array}{cl} \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{array}\right) & \begin{array}{l} \\ \\ \\ \leftarrow \text{row } s \\ \\ \\ \\ \leftarrow \text{row } t \\ \\ \\ \\ \end{array} \end{array}$$

- Consider the linear program

$$\min \quad c^T f \qquad\qquad c = \big(c(e_i)\big)_{1 \le i \le |E|}$$

$$\text{s.t.} \quad A \cdot f = b \qquad\qquad b =$$

$$f \geq 0.$$

If it has an optimal solution, then it also has an optimal solution with $f \in \{0,1\}^{|E|}$, and the simplex algorithm finds such a solution.

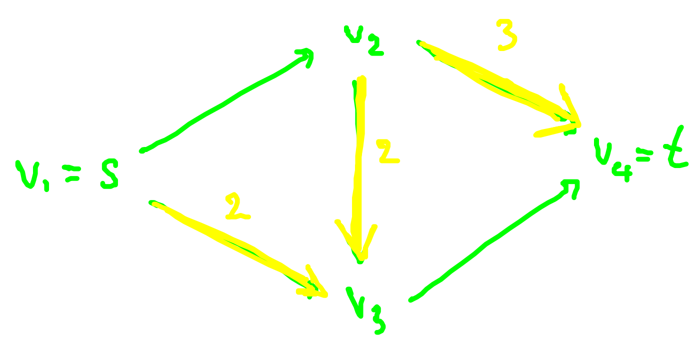
- Example: $\min \ (1,2,2,3,1) \cdot f$

$$\text{s.t.} \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 1 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{pmatrix} \cdot f = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

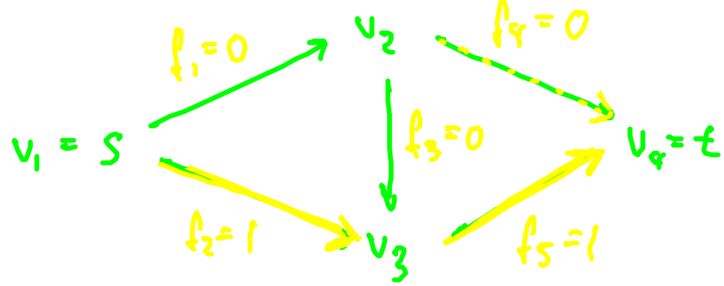
$$f_1, \ldots, f_5 \geq 0$$

bosis: 2, 3, 4

			f_2	f_3	f_4	
	3	1	0	0	0	0
$f_2 =$	1	1	1	0	0	0
$f_3 =$	-1	-1	0	1	0	-1
$f_4 =$	1	0	0	0	1	1



new basis: 2, 5, 4



	3	1	0	0	0	0
$f_2=$	1	1	1	0	0	0
$f_5=$	1	1	0	-1	0	1
$f_4=$	0	-1	0	1	1	0

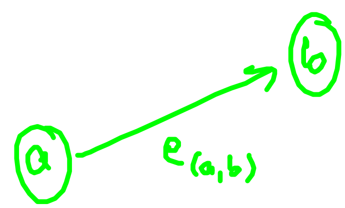
$\Rightarrow$ shortest path over edges e_2 and e_5 with cost 3

- dual problem:

$$\max \quad b^T p$$
$$\text{s.t.} \quad A^T \cdot p \leq c$$
$$p \text{ free}$$

interpretation:

$$\max \quad p_s - p_t$$
$$\text{s.t.} \quad p_a - p_b \leq c(e_{(a,b)})$$



objective function: $p_s - p_t$

$$= \underbrace{(p_s - p_{i_1})}_{\leq c(e_{j_1})} + \underbrace{(p_{i_1} - p_{i_2})}_{\leq c(e_{j_2})} + p_{i_2} \ldots - p_{i_k} + \underbrace{(p_{i_k} - t)}$$

$$\leq \sum_{e \in W} c(e)$$

$$W: \quad s \xrightarrow{e_{j_1}} v_{i_1} \xrightarrow{e_{j_2}} \cdots \longrightarrow v_{i_k} \longrightarrow t$$

$\longrightarrow$ dual program maximises the lower bound on the costs of paths between s and t

- complementary slackness: for p and f optimal dual/primal solution
  - $p_i \cdot \left(\sum_j a_{ij} f_j - b_i \right) = 0$ $\longrightarrow$ ✓ (std. form!)
  - $f_j \cdot \left(c_j - \sum_i p_i a_{ij} \right) = 0$

  $\longrightarrow$ if an edge e_j lies on a shortest path ($f_j > 0$), then it is pulled tight ($\sum_i p_i a_{ij} = c$); if it is not tight, then it cannot lie on the shortest path

- shadow prices ...