Fourier-Motzkin, Farkas-Lemma & Duality

- Problem: Decide whether a given polyhedron $Ax \geq b$ is nonempty.

$$\begin{array}{rrrrl}
x_1 & & -\ x_3 & \geq -1 & \quad \mathrm{I} \\
4x_1 & +9x_2 & -\ x_3 & \geq -7 & \quad \mathrm{II} \\
-8x_1 & +3x_2 & +2x_3 & \geq -7 & \quad \mathrm{III} \\
-5x_1 & -6x_2 & -4x_3 & \geq -7 & \quad \mathrm{IV} \\
x_1 & -3x_2 & +5x_3 & \geq -7 & \quad \mathrm{V}
\end{array}$$

$$\rightsquigarrow$$

$$\begin{array}{lrrl@{\qquad\qquad}rrl}
2\cdot\mathrm{I} + \mathrm{III} & -6x_1 & +3x_2 & \geq -9 & -2x_1 & +\ x_2 & \geq -3 \\
5\cdot\mathrm{I} + \mathrm{IV} & 6x_1 & -3x_2 & \geq -12 & 2x_1 & -\ x_2 & \geq -4 \\
2\cdot\mathrm{II} + \mathrm{III} & & 21x_2 & \geq -21 & & +\ x_2 & \geq -1 \\
5\cdot\mathrm{II} + \mathrm{V} & 21x_1 & +42x_2 & \geq -42 & x_1 & +2x_2 & \geq -2 \\
\mathrm{IV} + 2\cdot\mathrm{III} & -21x_1 & & \geq -21 & -x_1 & & \geq -1 \\
5\cdot\mathrm{IV} + 4\cdot\mathrm{V} & -21x_1 & -42x_2 & \geq -63 & -x_1 & -2x_2 & \geq -3
\end{array}$$

$$\rightsquigarrow$$

$$\begin{array}{rl@{\qquad\qquad}l}
-x_1 & \geq -1 & x_1 \leq 1 \\
0 & \geq -7 \ \checkmark & \\
-5x_1 & \geq -9 & (\ x_1 \leq 9/5\ ) \\
2x_1 & \geq -5 & (\ x_1 \geq -\frac{5}{2}\ ) \\
-x_1 & \geq -5 & (\ x_1 \leq 5\ ) \\
5x_1 & \geq -10 & x_1 \geq -2 \\
0 & \geq -5 \ \checkmark &
\end{array}$$

feasible!

- in general: Iteration of Fourier-Motzkin-Elimination:

Given $P = \{ x \in \mathbb{R}^n \mid a_i^T x \geq b_i \}$, $a_i \in \mathbb{R}^n$

$\mathcal{P} := \{ i \mid a_{in} > 0 \}$, $\mathcal{Z} := \{ i \mid a_{in} = 0 \}$, $\mathcal{N} := \{ i \mid a_{in} < 0 \}$

$Q \subseteq \mathbb{R}^{n-1}$ polyhedron defined by the following inequalities:

- $\bar{a}_i^T x \geq b_i$ for all $i \in \mathcal{Z}$
- $(a_{jn} \cdot \bar{a}_i^T + a_{in} \cdot \bar{a}_j^T) \cdot x \geq a_{jn} b_i + a_{in} b_j$ for all $(i,j) \in \mathcal{P} \times \mathcal{N}$

(where $\bar{a}_i^T = (a_1, \dots, a_{n-1})$)

<u>Thm.</u> $Q = \pi_{n-1}(P)$, the projection of P onto the subspace defined by the variables $x_1, \dots, x_{n-1}$

<u>Corollary.</u> Projections of polyhedra are again polyhedra.

- We can solve linear programs with Fourier-Motzkin:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t.}\ & Ax \geq b \end{aligned}$$

Trick: rewrite $c^T x$ as

$$x_0 := c^T x \iff x_0 - c^T x = 0 \iff \begin{aligned} x_0 - c^T x &\geq 0 \quad \wedge \\ -x_0 + c^T x &\geq 0 \end{aligned}$$

$$\Big\downarrow$$

$$\begin{aligned} \min\ & x_0 \\ \text{s.t.}\ & \left(\begin{array}{c|c} 1 & -c^T \\ -1 & +c^T \\ \hline 0 & \\ \vdots & A \\ 0 & \end{array}\right) \cdot \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ \hline b \end{pmatrix} \end{aligned}$$

$$\Big\downarrow \text{Fourier-Motzkin}$$

$$\begin{aligned} \min\ & x_0 \\ \text{s.t.}\ & \begin{pmatrix} A' \end{pmatrix} \cdot x_0 \geq \begin{pmatrix} b' \end{pmatrix} \end{aligned}$$

(easy to solve!)

Duality Thm. $\rightleftarrows$ Farkas' Lemma

$\Uparrow$ Simplex method with anticycling $\Uparrow$ Fourier-Motzkin

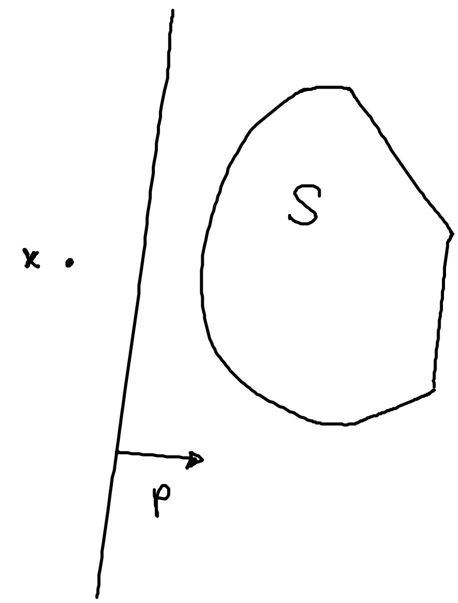
▸ <u>Seperating Hyperplane Theorem.</u>

$S \subseteq \mathbb{R}^n$ convex, non-empty, closed

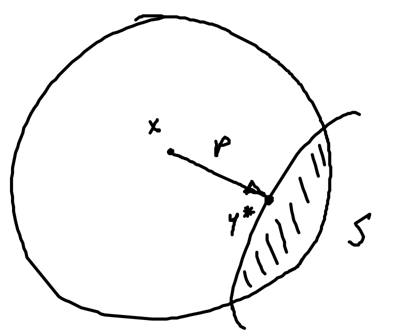
$x \notin S$.

Then there exists $p \in \mathbb{R}^n$ such that

$$p^T x < p^T y \quad \text{for all } y \in S.$$



Proof (Picture):



$p := y^* - x$

<u>Farkas' Lemma.</u> $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

Then <u>either</u> $A \cdot x = b$, $x \geq 0$ is feasible

<u>or</u> there exists a $p \in \mathbb{R}^m$ such that $p^T A \geq 0$, $p^T b < 0$.

In other words:

$$Ax = b,\ x \geq 0 \ \text{infeasible} \iff \exists p \in \mathbb{R}^m : p^T \cdot A \geq 0,\ p^T \cdot b < 0$$

Proof: „$\Leftarrow$": $x \geq 0$, $Ax = b$, suppose $p^T \cdot A \geq 0$

$$\Rightarrow 0 > p^T b = p^T A x \geq 0 \quad \Rightarrow \lightning$$

„$\Rightarrow$": $S := \{ Ax \mid x \geq 0 \}$, suppose $b \notin S$

S is convex (easy), $S \neq \emptyset$ $(0 \in S)$, S is closed:

consider $S' := \{ (y, x) \mid y = Ax,\ x \geq 0 \}$ then

$S = \pi_m(S')$ and S is a polyhedron by the corollary to Fourier-Motzkin

$\overset{\text{Sep. Hyp. Thm.}}{\Rightarrow} \quad \exists p: \quad p^T b < p^T y \quad \forall y \in S$

$\Rightarrow \quad p^T b < 0 \qquad$ since $\quad 0 \in S$

Let A_i be a column of A

$\Rightarrow \lambda \cdot A_i \in S \quad \forall \lambda > 0$

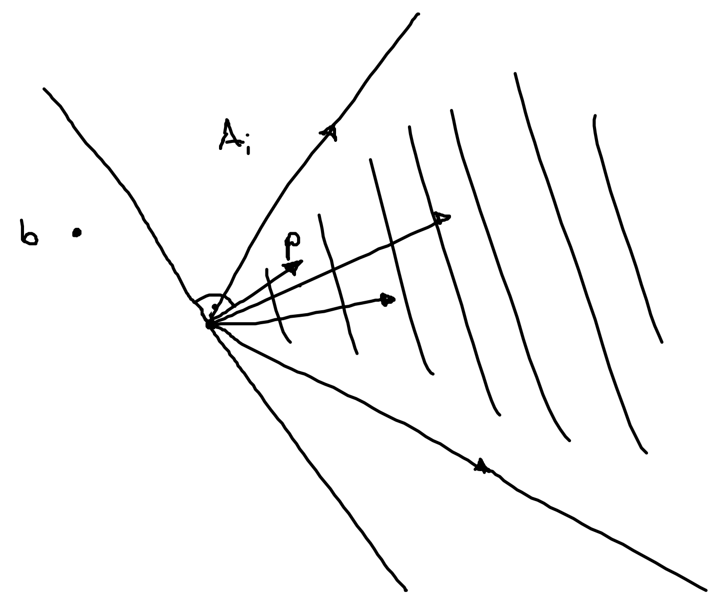
$\Rightarrow p^T b < \lambda p^T A_i \quad \forall \lambda > 0$

$\Rightarrow p^T A_i > \frac{1}{\lambda} \cdot p^T b \xrightarrow{\lambda \to \infty} 0$

$\Rightarrow p^T A_i \geq 0$

$\square$

▸ Picture:



▸ equivalent statement: $A_1, \ldots, A_n, b$ given and

p with $p^T A_i \geq 0 \ \forall i$ implies $p^T b \geq 0$

$$\Longrightarrow b = \sum A_i x_i \quad \text{with } x_i \geq 0$$

Duality Theorem.

$$(P) \quad \begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax \geq b \end{array} \qquad\qquad (D) \quad \begin{array}{ll} \max & p^T b \\ \text{s.t.} & p^T A = c^T \\ & p \geq 0 \end{array}$$

If (P) has an optimal solution x^*, then (D) has an optimal solution p^*

and $\quad c^T x^* = p^{*T} b$.

<u>Proof</u>: $\quad I := \{ i \mid a_i^T x^* = b_i \}$

Let d, $a_i^T d \geq 0 \quad \forall i \in I$

This implies: $\quad c^T d \geq 0$.

$\overset{\text{Farkas}}{\Longrightarrow} \quad c = \sum_{i \in I} p_i \cdot a_i \qquad$ with $\quad p_i \geq 0$

Set $p_i = 0 \quad$ for all $i \notin I$

Then $\quad p^T A = c^T \qquad$ (p is a feasible solution)

and $\quad p^T b = \sum_{i \in I} p_i b_i = \sum_{i \in I} p_i a_i^T x^* = c^T x^*$

$\overset{\text{weak duality}}{\Longrightarrow} \quad p$ is optimal $\qquad \square$.