

Fourier-Motzkin, Farkas-Lemma & Duality

► Problem: Decide whether a given polyhedron $Ax \geq b$ is nonempty.

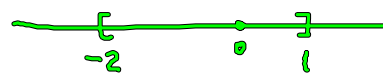
$$\begin{array}{rcl}
 x_1 & - x_3 & \geq -1 & \text{I} \\
 4x_1 + 9x_2 - x_3 & & \geq -7 & \text{II} \\
 -8x_1 + 3x_2 + 2x_3 & & \geq -7 & \text{III} \\
 -5x_1 - 6x_2 - 4x_3 & & \geq -7 & \text{IV} \\
 x_1 - 3x_2 + 5x_3 & & \geq -7 & \text{V}
 \end{array}$$



$$\begin{array}{rcl}
 2 \cdot \text{I} + \text{III} & -6x_1 + 3x_2 & \geq -9 & -2x_1 + x_2 \geq -3 \\
 5 \cdot \text{I} + \text{IV} & 6x_1 - 3x_2 & \geq -12 & 2x_1 - x_2 \geq -4 \\
 2 \cdot \text{II} + \text{III} & & 21x_2 \geq -21 & + x_2 \geq -1 \\
 5 \cdot \text{II} + \text{V} & 21x_1 + 42x_2 & \geq -42 & x_1 + 2x_2 \geq -2 \\
 \text{IV} + 2 \cdot \text{III} & -21x_1 & \geq -21 & -x_1 \geq -1 \\
 5 \cdot \text{IV} + 4 \cdot \text{V} & -21x_1 - 42x_2 & \geq -63 & -x_1 - 2x_2 \geq -3
 \end{array}$$



$$\begin{array}{rcl}
 -x_1 & \geq -1 & x_1 \leq 1 \\
 0 & \geq -7 & \checkmark & (x_1 \leq 9/5) \\
 -5x_1 & \geq -9 & & (x_1 \geq -9/5) \\
 2x_1 & \geq -5 & & (x_1 \leq 5) \\
 -x_1 & \geq -5 & & \\
 5x_1 & \geq -10 & & x_1 \geq -2 \\
 0 & \geq -5 & \checkmark &
 \end{array}$$



feasible!

► in general: Iteration of Fourier-Motzkin-Elimination:

Given $P = \{x \in \mathbb{R}^n \mid a_i^T x \geq b_i\}$, $a_i \in \mathbb{R}^n$

$$\mathcal{P} := \{i \mid a_{in} > 0\}, \quad \mathcal{Z} := \{i \mid a_{in} = 0\}, \quad \mathcal{N} := \{i \mid a_{in} < 0\}$$

$Q \subseteq \mathbb{R}^{n-1}$ polyhedron defined by the following inequalities:

- $\bar{a}_i^T x \geq b_i$ for all $i \in \mathcal{Z}$
- $(a_{jn} \cdot \bar{a}_i^T + a_{in} \cdot \bar{a}_j^T) \cdot x \geq a_{jn} b_i + a_{in} b_j$ for all $(i, j) \in \mathcal{P} \times \mathcal{N}$

(where $\bar{a}_i^T = (a_{i1}, \dots, a_{i, n-1})$)

Thm. $Q = \pi_{n-1}(\mathcal{P})$, the projection of \mathcal{P} onto the subspace defined by the variables x_1, \dots, x_{n-1}

Corollary. Projections of polyhedra are again polyhedra.

► We can solve linear programs with Fourier-Motzkin:

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

Trick: rewrite $c^T x$ as

$$\begin{aligned} x_0 = c^T x &\iff x_0 - c^T x = 0 \\ &\iff x_0 - c^T x \geq 0 \quad \wedge \\ &\quad -x_0 + c^T x \geq 0 \end{aligned}$$

$$\begin{aligned} \min \quad & x_0 \\ \text{s.t.} \quad & \begin{pmatrix} 1 & -c^T \\ -1 & +c^T \\ 0 & A \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} \end{aligned}$$

$$\left. \begin{aligned} \min \quad & x_0 \\ \text{s.t.} \quad & \begin{pmatrix} 1 & -c^T \\ -1 & +c^T \\ 0 & A \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} \geq \begin{pmatrix} 0 \\ 0 \\ b \end{pmatrix} \end{aligned} \right\} \text{Fourier-Motzkin}$$

$$\min \quad x_0$$

$$\text{s.t.} \quad \begin{pmatrix} A' \end{pmatrix} \cdot x_0 \geq \begin{pmatrix} b' \end{pmatrix} \quad (\text{easy to solve!})$$

Duality Thm. \Leftrightarrow Farkas' Lemma



Simplex method
with anti-cycling



Fourier-Stolatin

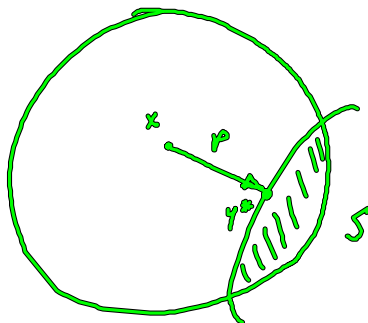
► Separating Hyperplane Theorem.

$S \subseteq \mathbb{R}^n$ convex, non-empty, closed
 $x \notin S$.

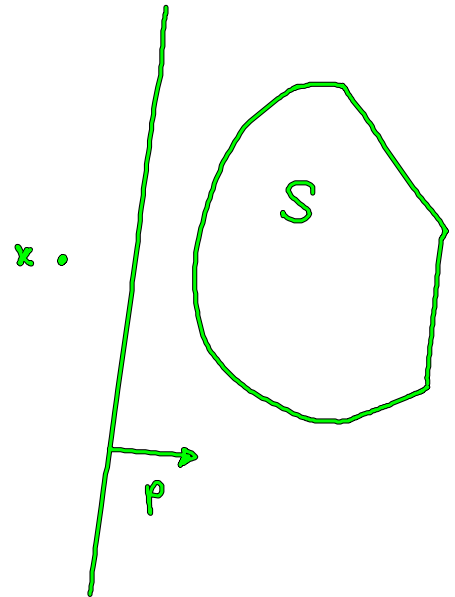
Then there exists $p \in \mathbb{R}^n$ such that

$$p^T x < p^T y \quad \text{for all } y \in S.$$

Proof (Picture):



$$p := y^* - x$$



Farkas' Lemma. $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

Then either $A \cdot x = b$, $x \geq 0$ is feasible

or there exists a $p \in \mathbb{R}^m$ such that $p^T A \geq 0$, $p^T b < 0$.

In other words:

$$Ax = b, x \geq 0 \text{ infeasible} \Leftrightarrow \exists p \in \mathbb{R}^m: p^T A \geq 0, p^T b < 0$$

Proof: " \Leftarrow ": $x \geq 0$, $Ax = b$, suppose $p^T A \geq 0$

$$\Rightarrow 0 > p^T b = p^T Ax \geq 0 \quad \rightarrow \zeta$$

" \Rightarrow ": $S := \{Ax \mid x \geq 0\}$, suppose $b \notin S$

S is convex (easy), $S \neq \emptyset$ ($0 \in S$), S is closed:

consider $S' := \{ (y, x) \mid y = Ax, x \geq 0 \}$ then

$S = \pi_m(S')$ and S is a polyhedron by the corollary to Farkas-Motzkin

Sep. Hyp. Thm.

$$\Rightarrow \exists p: p^T b < p^T y \quad \forall y \in S$$

$$\Rightarrow p^T b < 0 \quad \text{since } 0 \in S$$

Let A_i be a column of A

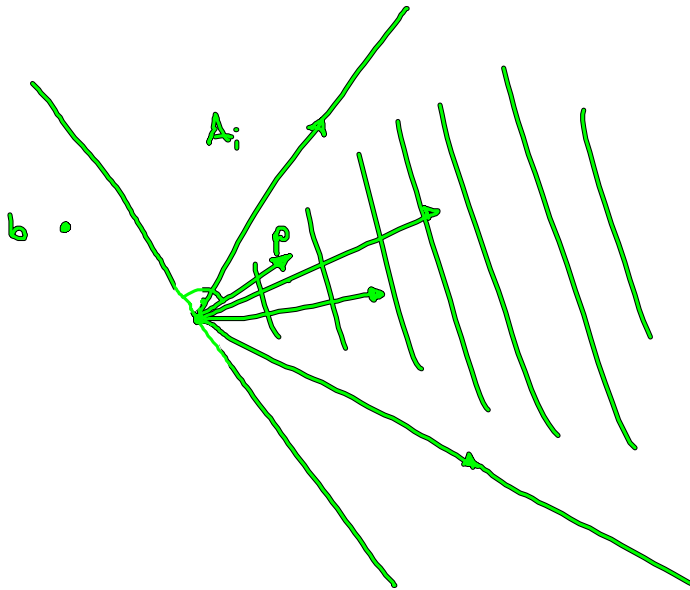
$$\Rightarrow \lambda \cdot A_i \in S \quad \forall \lambda > 0$$

$$\Rightarrow p^T b < \lambda p^T A_i \quad \forall \lambda > 0$$

$$\Rightarrow p^T A_i > \frac{1}{\lambda} \cdot p^T b \xrightarrow{\lambda \rightarrow \infty} 0$$

$$\Rightarrow p^T A_i \geq 0$$

► Picture:



► equivalent statement: A_1, \dots, A_n, b given and p with $p^T A_i \geq 0 \quad \forall i$ implies $p^T b \geq 0$

$$\Rightarrow b = \sum A_i x_i \quad \text{with } x_i \geq 0$$

Duality Theorem.

$$(P) \quad \min c^T x \\ \text{s.t. } Ax \geq b$$

$$(D) \quad \max p^T b \\ \text{s.t. } p^T A = c^T \\ p \geq 0$$

If (P) has an optimal solution x^* , then (D) has an optimal solution p^*

$$\text{and } c^T x^* = p^{*T} b$$

Proof: $I := \{i \mid a_i^T x^* = b_i\}$

$$\text{Let } d, a_i^T d \geq 0 \quad \forall i \in I$$

$$\text{This implies: } c^T d \geq 0$$

Farkas

$$\Rightarrow c = \sum_{i \in I} p_i \cdot a_i \quad \text{with } p_i \geq 0$$

$$\text{Set } p_i = 0 \quad \text{for all } i \notin I$$

$$\text{Then } p^T A = c^T \quad (p \text{ is a feasible solution})$$

$$\text{and } p^T b = \sum_{i \in I} p_i b_i = \sum_{i \in I} p_i a_i^T x^* = c^T x^*$$

weak duality

$$\Rightarrow p \text{ is optimal}$$

□