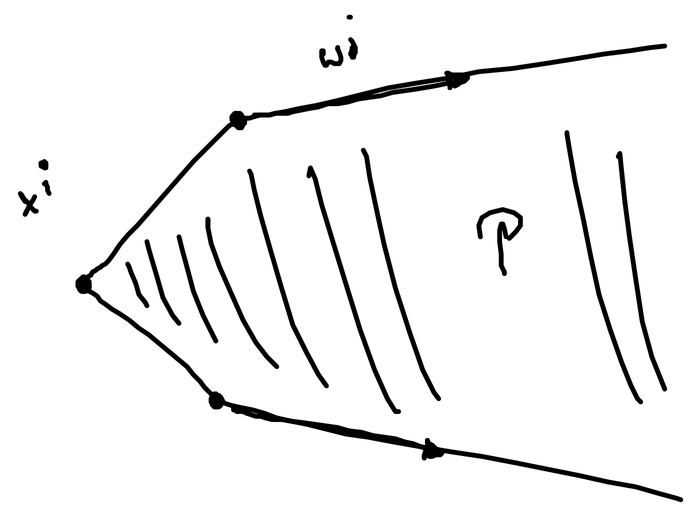
Resolution Theorem

$\mathcal{P} = \{ x \in \mathbb{R}^n \mid Ax \geq b \}$ with at least one extreme point

Then $\mathcal{P} = \{ \sum \lambda_i x^i + \sum \theta_j w^j \mid \theta_j, \lambda_i \geq 0, \sum \lambda_i = 1 \}$

where x^i are the extreme points
and w^j are the extreme rays of $\mathcal{P}$



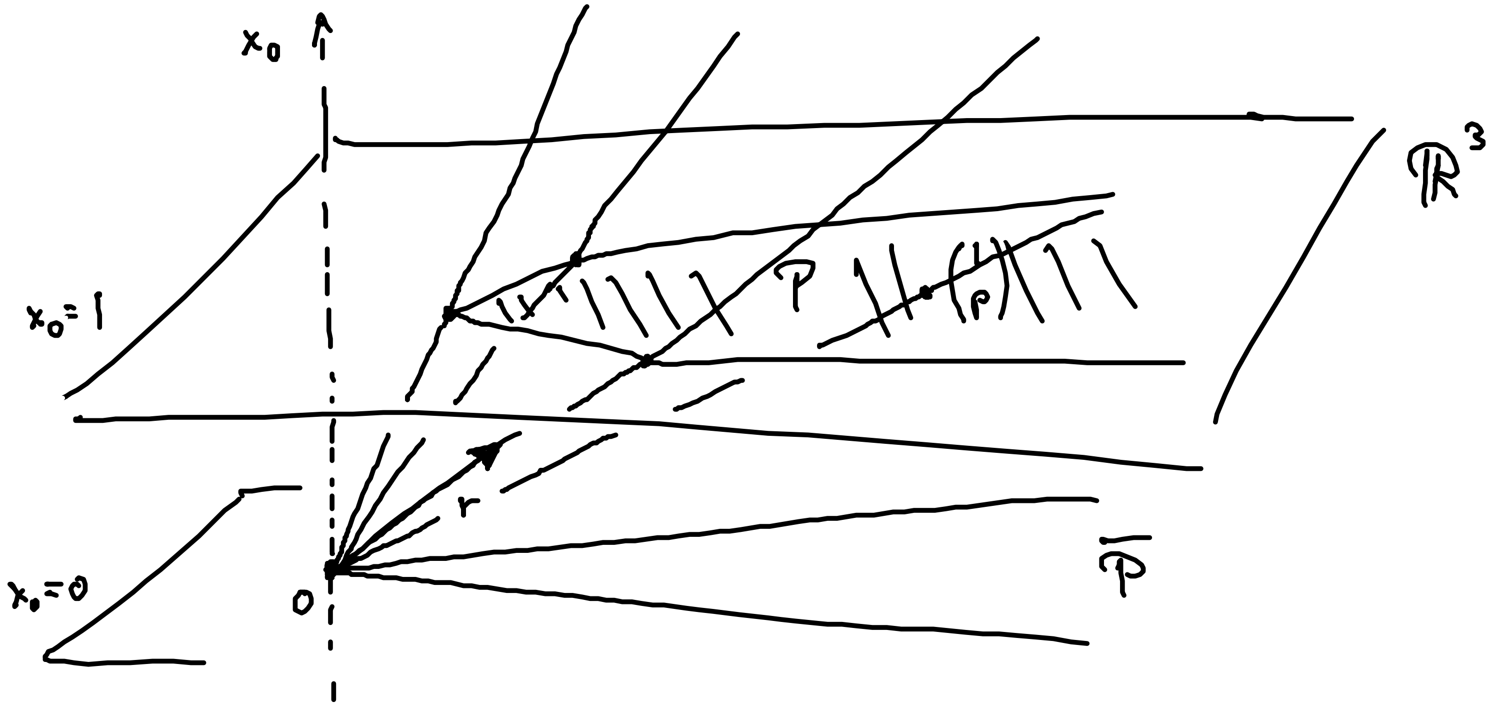
- lifting

(affine) inequality $\underset{\substack{\| \\ (a_1, \ldots, a_n)}}{a^T} x \geq b_i$ (i-th row)

$\longrightarrow$ (linear) inequality $-b_i x_0 + a_1 x_1 + \ldots + a_n x_n \geq 0$

$\overline{\mathcal{P}} := \{ x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^{n+1} \mid (-b \mid A) x \geq 0, \; x_0 \geq 0 \}$

$$\mathcal{P} = \overline{\mathcal{P}} \cap \{ x \in \mathbb{R}^{n+1} \mid x_0 = 1 \}$$



ray in $\mathbb{R}^{n+1}$: $\quad r = \left\{ \lambda \cdot \begin{pmatrix} r_0 \\ r_1 \\ \vdots \\ r_n \end{pmatrix} \;\middle|\; \lambda \geq 0 \right\}$

if $r_0 \neq 0 \longrightarrow$ normalise $\quad r = \left\{ \lambda \cdot \begin{pmatrix} 1 \\ r_1 \\ \vdots \\ r_n \end{pmatrix} \right\}$

extreme rays of $\overline{\mathcal{P}}$ give extreme $\left\{ \begin{array}{ll} \text{points} & \text{if } r_0 = 1 \\ \text{rays} & \text{if } r_0 = 0 \end{array} \right\}$ of $\mathcal{P}$

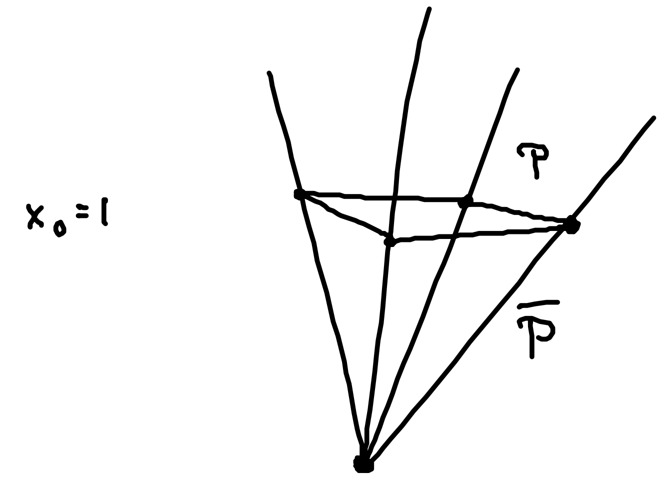
$$p \in \mathcal{P}: \quad \begin{pmatrix} 1 \\ p \end{pmatrix} = \sum \mu_i \underbrace{r^i}_{\text{extreme rays of } \overline{\mathcal{P}}} = \sum \mu_i \begin{pmatrix} 1 \\ r_1^i \\ \vdots \\ r_n^i \end{pmatrix} + \sum \mu_i \begin{pmatrix} 0 \\ r_1^i \\ \vdots \\ r_n^i \end{pmatrix}$$

$$\longleftrightarrow \quad p = \sum \lambda_i x^i + \sum \theta_i w^i$$

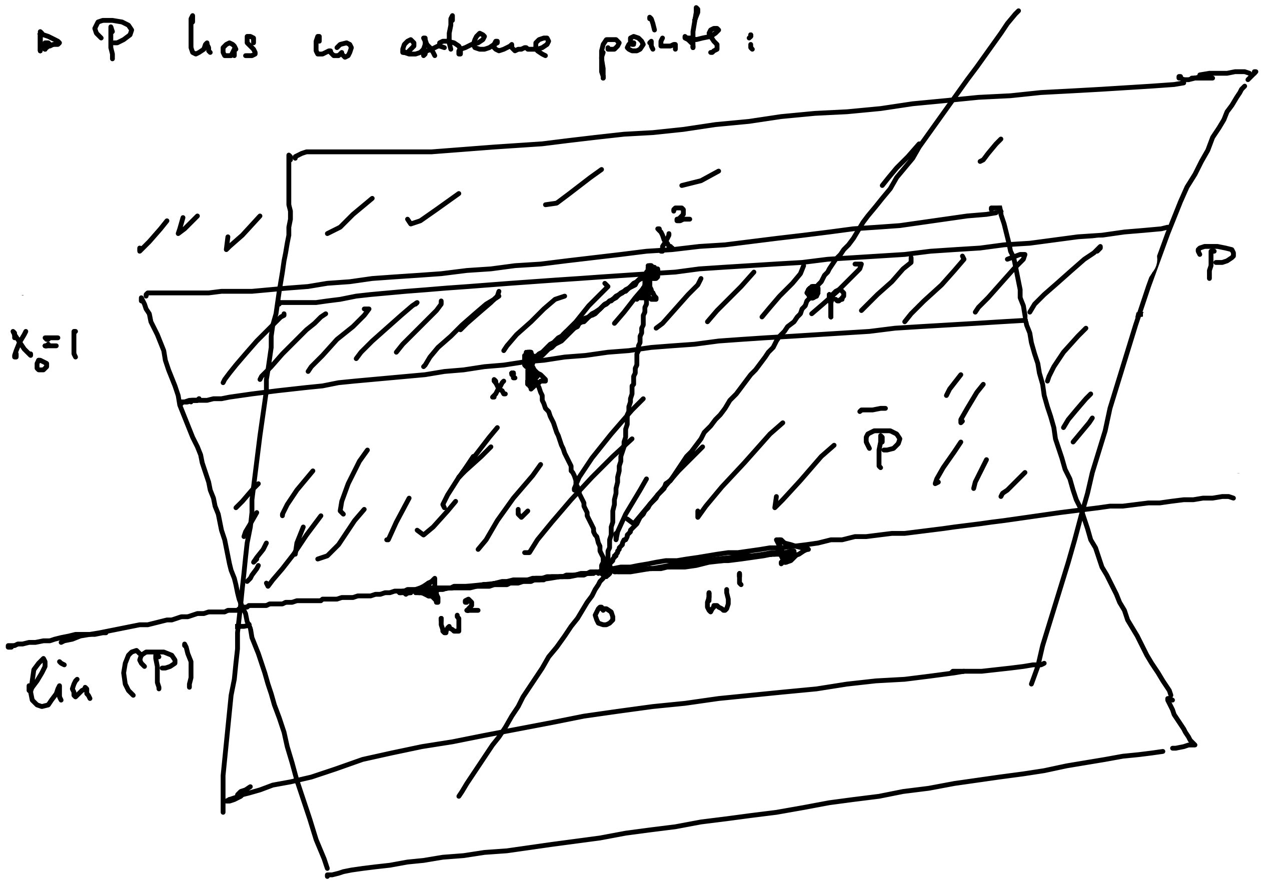
$$1 = \sum \lambda_i$$

(res. thm.)

- P has no extreme rays:



- P has no extreme points:



$$\binom{1}{p} = \sum \mu_i \binom{1}{r^i} = \sum \lambda_i \binom{1}{x^i} + \sum \theta_i \binom{0}{w^i}$$

$\longrightarrow \quad p = \sum \lambda_i x^i + \sum \theta_i w^i$

$1 = \sum \lambda_i$

$\longrightarrow$ $\mathcal{P}$ can be written as the sum of a pointed polyhedron and a subspace $\operatorname{lin}(\mathcal{P})$

$$\mathcal{P} = \underbrace{\operatorname{lin}(\mathcal{P})}_{} + \left(\mathcal{P} \cap \operatorname{lin}(\mathcal{P})^{\perp}\right)$$

$= \langle l_1, \dots, l_k \rangle$

where $l_1, \dots, l_k$ are the lines contained in $\mathcal{P}$

... projective space: $\mathbb{R}P^n$ $\qquad \mathbb{F}_2 P^n$

$\mathbb{R}P^n := \left\{ \{\lambda r \mid \lambda \in \mathbb{R}\} \;\middle|\; r \in \mathbb{R}^{n+1} \setminus \{0\} \right\}$ the set of all

1-dimensional subspaces of $\mathbb{R}^{n+1}$ ("points")

"lines" in $\mathbb{R}P^n$ are the 2-dim. subspaces of $\mathbb{R}^{n+1}$