

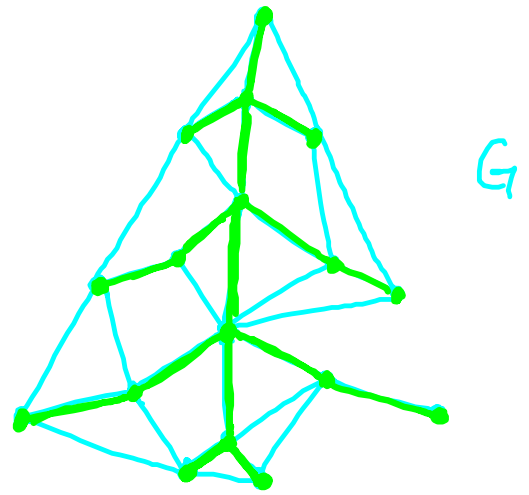
# MST



Graph  $G = (V, E)$ , undirected,  
connected, costs  $c_e \geq 0 \quad \forall e \in E$

A spanning tree is a subgraph

$T = (V, F)$ ,  $F \subseteq E$ ,  
which is a tree (i.e.  $T$  is  
cycle-free & connected).



A minimal spanning tree (MST) is a spanning tree

$T$  with minimal cost  $\sum_{e \in F} c_e$ .

► LP:  $\min \sum_{e \in E} c_e \cdot x_e$

s.t.  $\sum_{e \in E} x_e = |V| - 1$

$\sum x_e \leq |S| - 1 \quad \forall \emptyset \neq S \subsetneq V$

$$e \in \mathcal{g}(S) \quad x_e \geq 0$$

$$\mathcal{g}(S) := \{e \in E \mid e = \{u, v\}, u, v \in S\}$$



→ LP gives a Lower bound on the cost of a MST.

i.e.: LP is a relaxation of MST.

► MST & TSP:

	MST	TSP
problem:	easy	NP-hard
LP relaxation:	exact	yields only lower bound
solve LP:	difficult (exponentially many constraints!)	by cutting plane method

► Let  $T = (V, F)$  be a minimal spanning tree for  $G = (V, E)$  and  $x^* = (x_e^*)_{e \in E}$  with

$$x_e^* = \begin{cases} 1 & \text{if } e \in F \\ 0 & \text{if } e \notin F \end{cases}. \quad \text{Then } x^* \text{ is an optimal solution of (LP).}$$

Proof.

# i. Reformulation of (LP)

$$(LP') \quad \min \sum_{e \in E} c_e x_e \quad \Leftrightarrow \quad \max \sum_{e \in E} -c_e x_e$$

$$\text{s.t.} \quad \sum_{e \in E} x_e = |V| - 1$$

$$\sum_{e \in A} x_e \leq |V| - \kappa(A) \quad \forall A \subsetneq E$$

$$x_e \geq 0$$

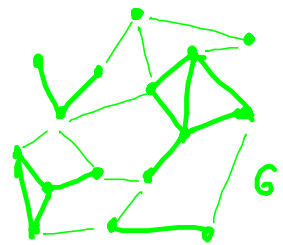
(LP)  $\vdots$   
 $\sum_{e \in \gamma(S)} x_e \leq |S| - 1 \quad \forall S$



where  $\kappa(A) = \#$  of connected components of the subgraph induced by  $A$

(LP) is equivalent to (LP')!

(ii) Dualise (LP')  $\rightarrow$  (DP')



$$\min \sum (|V| - \kappa(A)) \gamma_A$$

$$\text{s.t.} \quad \sum_{A \ni e} \gamma_A \geq -c_e \quad \forall e \in E$$

$$\gamma_A \geq 0 \quad \forall A \subsetneq E$$

$$\gamma_E \text{ free}$$

(iii) Suppose we have a solution  $y^*$  of (DP') that satisfies complementary slackness conditions together with  $x^*$ .

Then by compl. slack. both  $x^*$  and  $y^*$  are optimal for (LP'), (DP') resp.

(iv) Construct such a  $y^*$  by Kruskal's algorithm:

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### Kruskal

Input: Graph  $G = (V, E)$

Output: MST  $T = (V, F)$  for  $G$

1. Order the edges  $E = \{e_1, \dots, e_m\}$   
such that  $c_{e_1} \leq c_{e_2} \leq \dots \leq c_{e_m}$ .

2.  $T := (V, F)$  with  $F := \emptyset$

3. For  $i=1$  to  $m$

    If  $e_i$  connects different components of  $T$

        Then  $F = F \cup \{e_i\}$

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→ dual solution  $y^*$ :

$R_i := \{e_1, \dots, e_i\}$  (in the ordering of Kruskal's a.)

$1 \leq i \leq m$

$$\text{def } y_A^* := \begin{cases} c_{e_{i+1}} - c_{e_i} & \text{if } A = R_i, \quad 1 \leq i < m \\ -c_{e_m} & \text{if } A = R_m = E \\ 0 & \text{if } A \neq R_i \quad \forall i \end{cases}$$

•  $y^*$  is a solution of (DP') (order of  $e_i$ 's !)

$$\sum_{e_k \in A} y_A^* = \sum_{i=k}^m y_{R_i}^* = (c_{e_{k+1}} - c_{e_k}) + (c_{e_{k+2}} - c_{e_{k+1}}) \dots - c_{e_m} \\ = -c_{e_k}$$

$$y_A^* > 0$$

$$\Rightarrow \sum_{e \in A} x_e^* = |V| - \chi(A)$$

$\Rightarrow$  complementary slackness

□.