

Exponentially many constraints

► The ellipsoid method shows that solving a LP can always be done in time polynomial in the input size.

Even better: sometimes an LP can be solved in time polynomial in the number of variables, even if there are exponentially many constraints.

► Separation problem: Given a vector x

either assert that x is feasible

or find a violated constraint

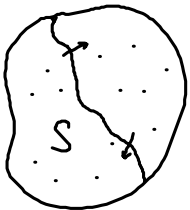
► TSP as linear program: $K_n = (V, E)$, costs $c_e \in \mathbb{R} \forall e \in E$

$$\min \sum_{e \in E} c_e x_e$$

$$\text{s.t. } \sum_{e \in \delta(v)} x_e = 2 \quad \forall v \in V \quad (n \text{ equalities})$$

$$\sum_{e \in \delta(S)} x_e \geq 2 \quad \forall S \subset V, S \neq \emptyset, V \quad (2^n - 2 \text{ inequalities})$$

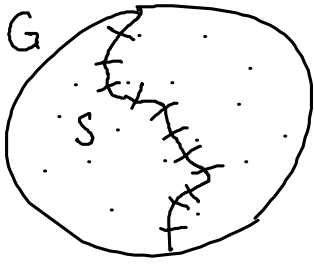
$$x_e \geq 0 \quad \forall e \in E \quad (n(n-1) \text{ variables})$$



Separation problem: Given $(x_e)_{e \in E} \geq 0$ with $\sum_{e \in \delta(v)} x_e = 2 \quad \forall v \in V$

either find an $S \subset V, S \neq \emptyset, V$ with $\sum_{e \in \delta(S)} x_e < 2$

or prove that no such S exists



→ try to find the S with

$$\text{minimal } \sum_{e \in \delta(S)} x_e$$

→ if this is $\geq 2 \Rightarrow$ all inequality constraints are satisfied

→ otherwise: we found such a set S !

→ Separation problem is a min-cut problem in G with x_e as edge capacities

and this is dual to the max-flow problem

► max flow & min cut: $G = (V, E)$ directed with edge capacities $u_e \forall e \in E$

choose $s, t \in V, s \neq t$

→ find a flow $(f_e)_{e \in E}$ that does not exceed the edge capacities and maximizes its value

→ LP:

$$\max \sum_{e \in \delta^+(s)} f_e - \sum_{e \in \delta^-(s)} f_e$$

$$\text{s.t. } \sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e = 0 \quad \forall v \in V \setminus \{s, t\} \quad (\text{flow conservation})$$

$$f_e \leq u_e \quad \forall e \in E$$

$$f_e \geq 0 \quad \text{---}$$

↓ dualize

$$\text{dual variables: } p_v \text{ free } \quad \forall v \in V \setminus \{s, t\}$$

$$q_e \geq 0 \quad \forall e \in E$$

$$\min \sum_{e \in E} u_e q_e$$

$$\text{s.t. } p_u - p_v + q_e \geq 0 \quad \forall e \in E, e = (u, v), u, v \neq \{s, t\}$$

$$-p_v + q_e \geq 1 \quad \forall e = (s, v)$$

$$p_u + q_e \geq -1 \quad \forall e = (u, s)$$

$$p_u + q_e \geq 0$$

$$\forall e = (u, t)$$

$$-p_v + q_e \geq 0$$

$$\forall e \in (t, v)$$

rewrite

introduce $p_s = -1$
 $p_t = 0$

$$\min \sum_{e \in E} u_e q_e$$

$$\text{s.t. } p_u - p_v + q_e \geq 0$$

$$\forall e = (u, v) \in E$$

$$p_s = 0$$

$$p_t = 1$$

$$p_v \text{ free}$$

$$q_e \geq 0$$

$$\forall v \in V \setminus \{s, t\}$$

$$\forall e \in E$$

► this really calculates a minimal cut ! :

general observation: Every feasible flow has value at most the capacity of any s-t-cut.

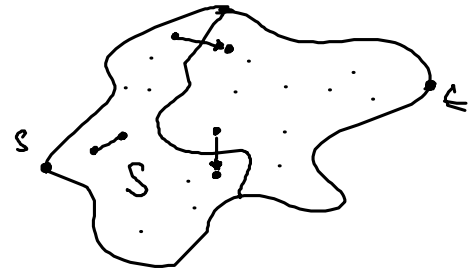
Let S be an s-t-cut and $(f_e)_{e \in E}$ a feasible flow

$$\Rightarrow \sum_{e \in \delta^+(S)} f_e - \sum_{e \in \delta^-(S)} f_e$$

$$= \sum_{v \in S} \left(\sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e \right)$$

$$= \sum_{e \in \delta^+(S)} f_e - \sum_{e \in \delta^-(S)} f_e \leq \sum_{e \in \delta^+(S)} u_e$$

$\uparrow \leq u_e$ $\uparrow \geq 0$



□

Let f^*, p^*, q^* be an optimal primal / dual solution.

$$S := \{v \in V \mid p_v^* < 1\} \Rightarrow s \in S, t \notin S$$

$$\Rightarrow \sum_{e \in \delta^+(S)} f_e^* - \sum_{e \in \delta^-(S)} f_e^* = \sum_{e \in \delta^+(S)} f_e^* - \sum_{e \in \delta^-(S)} f_e^* = \sum_{e \in \delta^+(S)} u_e \Rightarrow S \text{ is minimal}$$

$$e = (u, v) \in \delta^+(S) \Rightarrow q_e^* \geq p_v^* - p_u^* > 0 \Rightarrow f_e^* = u_e$$

$\begin{matrix} = u_e \\ \uparrow & \uparrow \\ \geq 1 & < 1 \end{matrix}$

$\begin{matrix} = 0 \\ \text{comp. sl.} \end{matrix}$

$$e = (u, v) \in \delta^-(S) \Rightarrow p_u^* - p_v^* + q_e^* \geq p_u^* - p_v^* > 0 \Rightarrow f_e^* = 0$$

$\begin{matrix} \uparrow & \uparrow \\ \geq 1 & < 1 \end{matrix}$

$\begin{matrix} \text{comp. sl.} \\ = 0 \end{matrix}$