

Exponentially many constraints

► The ellipsoid method shows that solving a LP can always be done in time polynomial in the input size.

Even better: sometimes an LP can be solved in time polynomial in the number of variables, even if there are exponentially many constraints.

► Separation problem: Given a vector x

either assert that x is feasible
or find a violated constraint

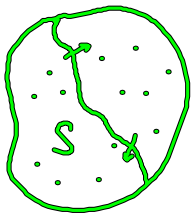
► TSP as linear program: $K_n = (V, E)$, costs $c_e \in \mathbb{R} \forall e \in E$

$$\min \sum_{e \in E} c_e x_e$$

$$\text{s.t. } \sum_{e \in \delta(v)} x_e = 2 \quad \forall v \in V \quad (n \text{ equalities})$$

$$\sum_{e \in \delta(S)} x_e \geq 2 \quad \forall S \subset V, S \neq \emptyset, V \quad (2^n - 2 \text{ inequalities})$$

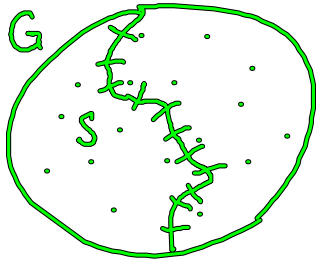
$$x_e \geq 0 \quad \forall e \in E \quad (n(n-1) \text{ variables})$$



Separation problem: Given $(x_e)_{e \in E} \geq 0$ with $\sum_{e \in \delta(v)} x_e = 2 \quad \forall v \in V$

either find an $S \subset V, S \neq \emptyset, V$ with $\sum_{e \in \delta(S)} x_e < 2$

or prove that no such S exists



→ try to find the S with

$$\text{minimal } \sum_{e \in S(S)} x_e$$

→ if this is $\geq 2 \Rightarrow$ all inequality constraints are satisfied

→ otherwise: we found such a set S !

→ Separation problem is a min-cut problem in G with x_e as edge capacities

and this is dual to the max-flow problem

► max flow & min cut: $G = (V, E)$ directed with edge capacities $u_e \forall e \in E$

choose $s, t \in V, s \neq t$

→ find a flow $(f_e)_{e \in E}$ that does not exceed the edge capacities and maximizes its value

→ LP:

$$\max \sum_{e \in S^+(s)} f_e - \sum_{e \in S^-(s)} f_e$$

$$\text{s.t. } \sum_{e \in S^+(v)} f_e - \sum_{e \in S^-(v)} f_e = 0 \quad \forall v \in V \setminus \{s, t\} \quad (\text{flow conservation})$$

$$f_e \leq u_e \quad \forall e \in E$$

$$f_e \geq 0 \quad \text{---}$$

↓
dualize

$$\text{dual variables: } p_v \text{ free } \quad \forall v \in V \setminus \{s, t\}$$

$$q_e \geq 0 \quad \forall e \in E$$

$$\min \sum_{e \in E} u_e q_e$$

$$\text{s.t. } p_u - p_v + q_e \geq 0 \quad \forall e \in E, e = (u, v), u, v \neq \{s, t\}$$

$$-p_v + q_e \geq 1 \quad \forall e = (s, v)$$

$$p_u + q_e \geq -1 \quad \forall e = (u, s)$$

$$\begin{array}{ll}
 p_u + q_e \geq 0 & \forall e = (u, t) \\
 -p_v + q_e \geq 0 & \forall e = (t, v) \\
 \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} \text{rewrite} & \text{introduce } p_s = -1 \\
 & p_t = 0
 \end{array}$$

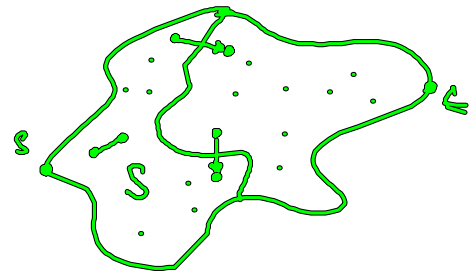
$$\begin{array}{ll}
 \min & \sum_{e \in E} u_e q_e \\
 \text{s.t.} & p_u - p_v + q_e \geq 0 \quad \forall e = (u, v) \in E \\
 & p_s = 0 \\
 & p_t = 1 \\
 & p_v \text{ free} \quad \forall v \in V \setminus \{s, t\} \\
 & q_e \geq 0 \quad \forall e \in E
 \end{array}$$

► this really calculates a minimal cut ! :

general observation: Every feasible flow has value at most the capacity of any s-t-cut.

Let S be an s-t-cut and $(f_e)_{e \in E}$ a feasible flow

$$\begin{aligned}
 & \Rightarrow \sum_{e \in \delta^+(S)} f_e - \sum_{e \in \delta^-(S)} f_e \\
 & = \sum_{v \in S} \left(\sum_{e \in \delta^+(v)} f_e - \sum_{e \in \delta^-(v)} f_e \right) \\
 & = \sum_{e \in \delta^+(S)} \underset{\leq u_e}{f_e} - \sum_{e \in \delta^-(S)} \underset{\geq 0}{f_e} \leq \sum_{e \in \delta^+(S)} u_e
 \end{aligned}$$



Let f^*, p^*, q^* be an optimal primal/dual solution.

$$S := \{v \in V \mid p_v^* < 1\} \quad \Rightarrow s \in S, t \notin S$$

$$\Rightarrow \sum_{e \in \delta^+(S)} f_e^* - \sum_{e \in \delta^-(S)} f_e^* = \sum_{e \in \delta^+(S)} \underset{\leq u_e}{f_e^*} - \sum_{e \in \delta^-(S)} \underset{\geq 0}{f_e^*} = \sum_{e \in \delta^+(S)} u_e \Rightarrow S \text{ is minimal}$$

$$e = (u, v) \in \delta^+(S) \Rightarrow q_e^* \Rightarrow p_v^* - p_u^* > 0 \Rightarrow f_e^* = u_e$$

$\begin{matrix} = u_e \\ \uparrow \\ \geq 1 \end{matrix} \quad \begin{matrix} \uparrow \\ < 1 \end{matrix}$

$$e = (u, v) \in \delta^-(S) \Rightarrow p_u^* - p_v^* + q_e^* \geq p_u^* - p_v^* > 0 \Rightarrow f_e^* = 0$$

$\begin{matrix} \uparrow \\ \geq 1 \end{matrix} \quad \begin{matrix} \uparrow \\ < 1 \end{matrix}$