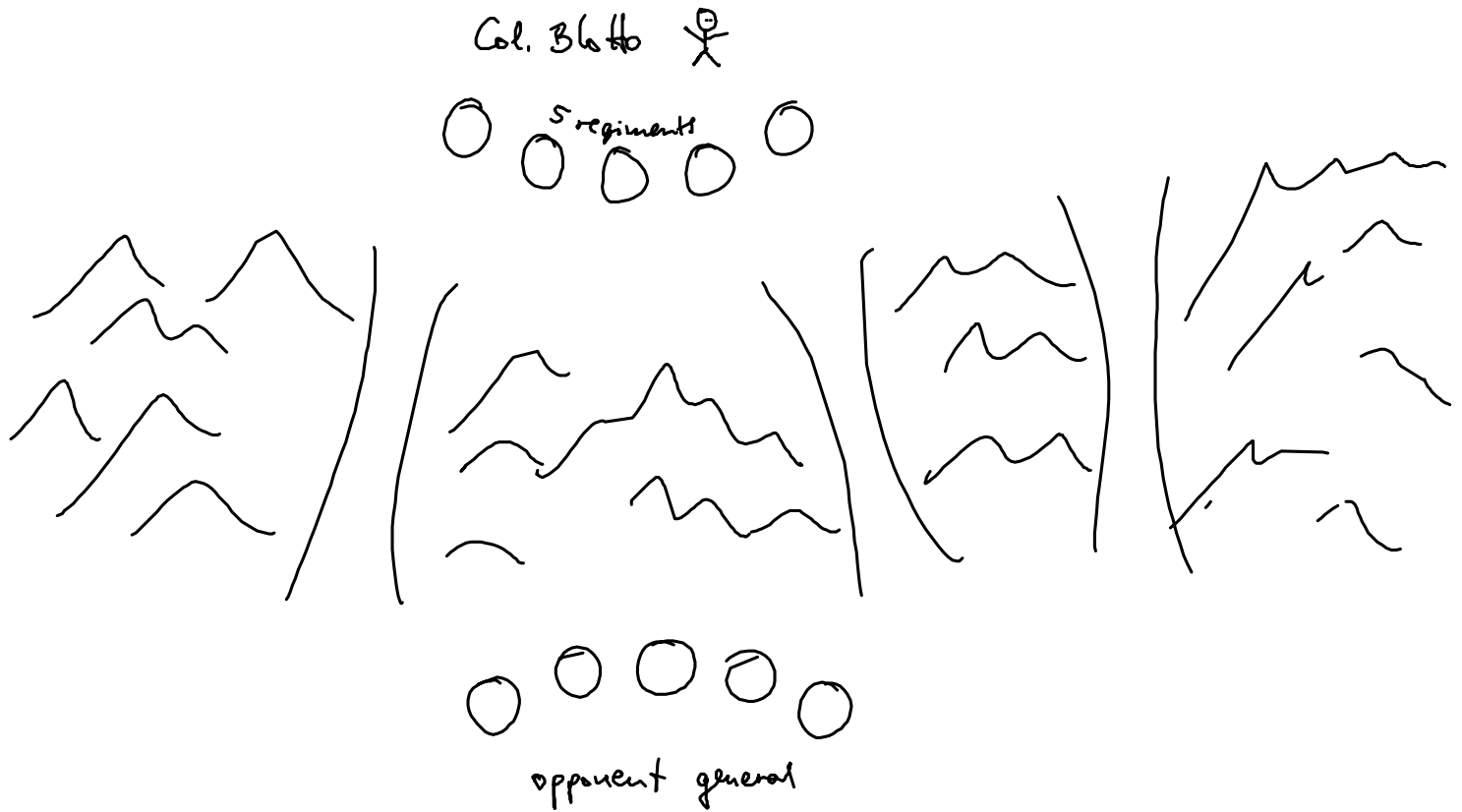


# Zero-sum matrix games

## ► „Game“ (Borel, 1921)



→ a mountain pass is won by the commander who sends more regiments to it than the other;  
the battle is won by the one who has conquered more passes than the other

→ partition the number of regiments:

$(5, 0, 0)$ ,  $(4, 1, 0)$ ,  $(3, 2, 0)$ ,  $(3, 1, 1)$ ,  $(2, 2, 1)$

assign them in one of  $3! = 6$  possible ways to the passes randomly

→ winning probabilities:

e.g. with  $(4, 1, 0)$  for Col. Blotto

and  $(5, 0, 0)$  for the opponent,

the winning probability for Col. Blotto is  $\frac{1}{3}$  and with prob.  $\frac{2}{3}$  there will be a draw  $\Rightarrow$  payoff:  $\frac{1}{3} - 0 = \frac{1}{3}$

→ payoff matrix (for Col. Blotto)

		Strategies of Col. Blotto				
		$(5, 0, 0)$	$(4, 1, 0)$	$(3, 2, 0)$	$(3, 1, 1)$	$(2, 2, 1)$
Strategies of Col. Blotto	→ $(5, 0, 0)$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	-1	-1
	→ $(4, 1, 0)$	$\frac{1}{3}$	0	0	$-\frac{1}{3}$	$-\frac{2}{3}$
	→ $(3, 2, 0)$	$\frac{1}{3}$	0	0	0	$\frac{1}{3}$
	→ $(3, 1, 1)$	1	$\frac{1}{3}$	0	0	$-\frac{1}{3}$
	→ $(2, 2, 1)$	1	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{1}{3}$	0







→ choose a strategy that guarantees the highest payoff in the worst case ( $\hat{=}$  smallest value in the row):  $(3, 2, 0)$

→ opponent: chooses a strategy that guarantees the lowest payoff in his worst case ( $\hat{=}$  highest value in the column):  $(3, 2, 0)$

→ strategies are best responses to each other  
(Nash equilibrium)

▶ another example: Rock, paper, scissors

→ payoff matrix: player B

				
Player A		0	-1	1
		1	0	-1
		-1	1	0

→ no (pure) Nash equilibrium!

→ solution: choose every strategy with some probability:  $\frac{1}{3}$

→ no reason to change the  $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ -strategy

→ mixed Nash equilibrium

↑  
mixed strategy

► In general: two players A and B with  $n$  and  $m$  pure strategies, resp.

The payoff matrix is  $M := (m_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}}$  where

$m_{ij}$  gives the payoff for A if strategy  $i$  is played against str.  $j$ .

A mixed strategy is a probability distribution over the pure strategies:

$$\text{for A: } x = (x_1, \dots, x_n) \geq 0, \quad \sum_{i=1}^n x_i = 1$$

$$\text{for B: } y = (y_1, \dots, y_m) \geq 0, \quad \sum_{j=1}^m y_j = 1$$

Given two mixed strat.  $x$  and  $y$ , the expected payoff for A is:

$$\begin{aligned} & \sum_{i,j} m_{ij} \cdot P_{x,y} [A \text{ plays } i, B \text{ plays } j] \\ &= \sum_{i,j} m_{ij} \cdot P_x [A \text{ plays } i] \cdot P_y [B \text{ plays } j] \end{aligned}$$

$$= \sum_{i,j} m_{ij} \cdot x_i \cdot y_j$$

$$= x^T M y$$

→ If Player A plays strategy  $x$ , the worst case payoff is

$$\min_y x^T M y$$

→ Player A plays the strategy  $\tilde{x}$  that maximises  $\min_y x^T M y$

→ Player B plays the strategy  $\tilde{y}$  that minimises  $\max_x x^T M y$

$$\text{Then: } \min_y \tilde{x}^T M y \leq \tilde{x}^T M \tilde{y} \leq \max_x x^T M \tilde{y}$$

Theorem. (Minimax Theorem (J.v. Neumann, 1926) for zero-sum matrix games)

For every  $n \times m$  payoff matrix  $M$  there exist mixed strategies  $\tilde{x}$  and  $\tilde{y}$  such that

$$\max_x \min_y x^T M y = \min_y \max_x x^T M y = \tilde{x}^T M \tilde{y} = \max_x x^T M \tilde{y} = \min_y \max_x x^T M y.$$

$(\tilde{x}, \tilde{y})$  is called a Nash equilibrium,  $\tilde{x}^T M \tilde{y}$  is the game value.

Proof. Define  $\tilde{x}$  and  $\tilde{y}$  as above.

B's best response to a given mixed strategy  $x$  of A is the solution of the linear program

$$\begin{aligned} \min \quad & x^T M y \\ \text{s.t.} \quad & \sum y = 1 \\ & y \geq 0 \end{aligned}$$

Let  $\beta(x)$  be the solution of this linear program.

A's task is to maximise  $\beta(x)$  over all his strategies  $x$

Dualize the above LP:

$$\begin{aligned} \max \quad & x_0 \\ \text{s.t.} \quad & M^T x - \mathbb{1} x_0 \geq 0 \end{aligned}$$

This also has optimal solution  $\beta(x)$  !

Extend this:

$$(*) \quad \begin{aligned} \max \quad & x_0 \\ \text{s.t.} \quad & M^T x - \mathbb{1} x_0 \geq 0 \\ & \sum_{i=1}^n x_i = 1 \\ & x_1, \dots, x_n \geq 0 \end{aligned}$$

The optimal solution to  $(*)$  is  $\tilde{x}$  with optimal value  $\tilde{x}_0$ .

Do the same for player B ... then  $\tilde{y}$  is the optimal solution to the lin. program

$$(**) \quad \begin{aligned} \min \quad & y_0 \\ \text{s.t.} \quad & M y - \mathbb{1} y_0 \leq 0 \\ & \sum_{j=1}^m y_j = 1 \\ & y_1, \dots, y_m \geq 0 \end{aligned}$$

$(*)$  and  $(**)$  are dual to each other

$$\Rightarrow \tilde{x}_0 = \tilde{y}_0 \quad (\text{by the duality theorem})$$

□.