Cutting plane algorithm

Given an (IP)

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \\ & x \text{ integer} \end{aligned}$$

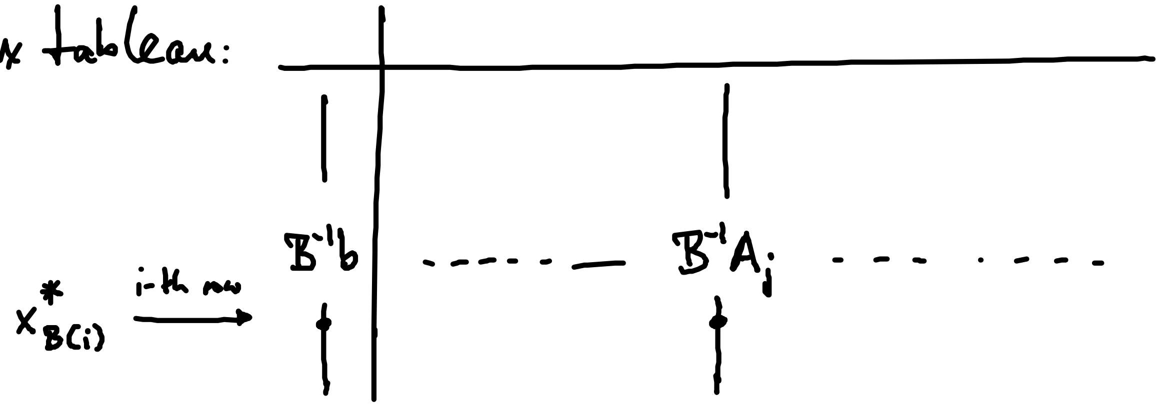
Algorithm:

1. Set up the linear programming relaxation (LP)
2. Solve (LP), let x^* be an optimal solution.
3. If x^* is integer, then STOP: x^* is optimal for (IP).
4. Otherwise add a new linear inequality to (LP) that is satisfied by all integer solution, but not by x^*; go to 2.

Gomory cuts

x^* opt. solution to the linear prog. relaxation

$x^* \notin \mathbb{Z}^n$, B basis matrix simplex tableau:



i-th row: $$x_{B(i)} + \sum_{j \in N} \overline{a_{ij}}\, x_j = \overline{a_{i0}}$$

with: $\overline{a_{ij}} := (B^{-1}A_j)_i$, $\overline{a_{i0}} := (B^{-1}b)_i$

N the set of indices of non-basic variables

Consider a row i where $\overline{a_{i0}}$ is fractional.

Then $$x^*_{B(i)} + \sum_{j \in N} \lfloor \overline{a_{ij}} \rfloor \underbrace{x^*_j}_{=0} = x^*_{B(i)} = \overline{a_{i0}} > \lfloor \overline{a_{i0}} \rfloor$$

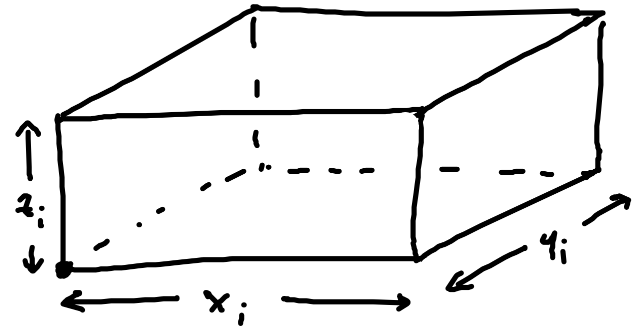
For all $\overset{\text{feasible}}{\text{integer}}$ solutions x :

$$x_{B(i)} + \sum_{j \in N} \underbrace{\lfloor \overline{a_{ij}} \rfloor}_{\leq \overline{a_{ij}}} \underbrace{x_j}_{\geq 0} \leq x_{B(i)} + \sum \overline{a_{ij}}\, x_j = \overline{a_{i0}}$$

$$\overset{x \text{ int.!}}{\Rightarrow} \boxed{x_{B(i)} + \sum_{j \in N} \lfloor \overline{a_{ij}} \rfloor x_j \leq \lfloor \overline{a_{i0}} \rfloor} \quad \text{Gomory cut}$$

Exercise 71.

Bricks given by their sizes



First ignore rotations.

- Variables: lower left front corner: $p_i^x,\ p_i^y,\ p_i^z \geq 0 \ \forall i$

  total box size: $s^x,\ s^y,\ s^z$

  → objective function: $\min\ s^x + s^y + s^z$

  some contraints:
$$\begin{aligned} p_i^x &+ x_i \leq s^x \\ p_i^y &+ y_i \leq s^y \\ p_i^z &+ z_i \leq s^z \end{aligned}$$

- non-overlapping ⟺ given a pair of bricks, one must be completely to the left or to the right of the other w.r.t. <u>at least</u> one coordinate

$$\Leftrightarrow \forall (i,j): \begin{array}{ccccccccc} p_i^x & + & x_i & \leq & p_j^x & \vee & p_j^x + x_j & \leq p_i^x & \vee \\ {}^y & & y & & {}^y & \vee & {}^y \ \ y & {}^y & \vee \\ {}^z & & z & & {}^z & \vee & {}^z \ \ z & {}^z & \end{array}$$

→ introduce 6 0/1-variables $\delta^{x/y/z}_{(i,j)}$, e.g. $\delta^{y}_{(j,i)} = 0$ means that $p^{y}_{j} + y_j \le p^{y}_{i}$ is satisfied

$$\Updownarrow$$

$$p^{y}_{i} - p^{y}_{j} - y_j \ge 0$$

$\delta^{y}_{(j,i)} = 1$ means: a trivial constraint is satisfied:

$$p^{y}_{i} - p^{y}_{j} - y_j \ge -M \cdot \delta^{y}_{(j,i)}$$

with $M := \sum_{i=1}^{n} x_i + y_i + z_i$

$\longrightarrow$ non-overlapping constraints:

$\forall (i,j):$

$$p^{x}_{j} - p^{x}_{i} - x_i \ge -M \cdot \delta^{x}_{(i,j)}$$

$$p^{y}_{j} - p^{y}_{i} - y_i \ge -M \cdot \delta^{y}_{(i,j)}$$

$$p^{z}_{j} - p^{z}_{i} - z_i \ge -M \cdot \delta^{z}_{(i,j)}$$

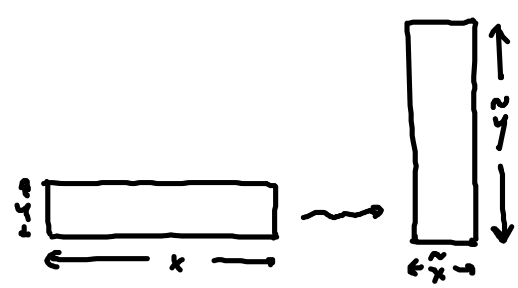
$$\delta^{x}_{(i,j)} + \delta^{x}_{(j,i)} + \delta^{y}_{(i,j)} + \delta^{y}_{(j,i)} + \delta^{z}_{(i,j)} + \delta^{z}_{(j,i)} \le 5$$

$$\delta^{x/y/z}_{(i,j)} \in \{0,1\}$$

- rotations

(for 2 dimensions)

If the brick is rotated then the sizes change:



$\sim$

$$\tilde{x} = y$$
$$\tilde{y} = x$$

$\longrightarrow$ decision variable $\sigma^{rot} \in \{0,1\}$
$\sigma^{non\,rot} \in \{0,1\}$

$\longrightarrow$ $\tilde{x} = \sigma^{non\,rot} x + \sigma^{rot} y$ $\qquad \sigma^{rot} + \sigma^{non\,rot} = 1$
$\tilde{y} = \sigma^{non\,rot} y + \sigma^{rot} x$

(in 3 dimensions)

$3! = 6$ different states corresponding to all permutations of the triple $[x, y, z)$

$\longrightarrow$ 6 0/1-variables σ_i^{π} $(\pi \in S_3)$ for every brick i

plus a constraint $\sum_{\pi \in S_3} \sigma_i^{\pi} = 1$

$$\longrightarrow \tilde{x}_i = \sigma_i^{id} x_i + \sigma_i^{(x,z,y)} x_i + \text{-----}$$
$$\tilde{y}_i = \sigma_i^{id} y_i + \sigma_i^{(x,z,y)} z_i + \text{----}$$
$$\tilde{z}_i = \sigma_i^{id} z_i + \sigma_i^{(x,z,y)} y_i + \text{-----}$$

$\longrightarrow$ use new sizes $\tilde{x}_i$, $\tilde{y}_i$, $\tilde{z}_i$ instead of x_i, y_i, z_i in all constraints