

Cutting plane algorithm

Given on (IP)

$$\begin{aligned} \text{min } & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0 \\ & x \text{ integer} \end{aligned}$$

Algorithm:

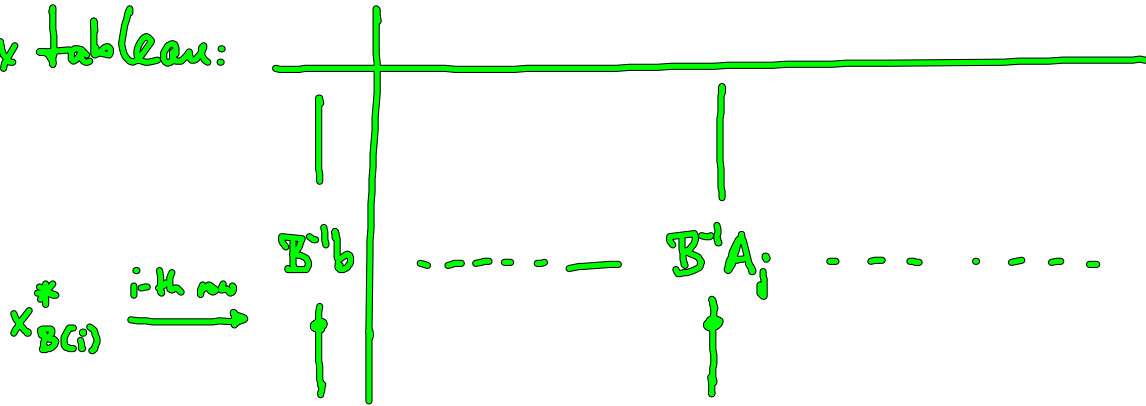
1. Set up the linear programming relaxation (LP)
2. Solve (LP), let x^* be an optimal solution.
3. If x^* is integer, then STOP: x^* is optimal for (IP).
4. Otherwise add a new linear inequality to (LP) that is satisfied by all integer solution, but not by x^* ; go to 2.

Gomory cuts

x^* opt. solution to the linear prog. relaxation

$x^* \notin \mathbb{Z}^n$, B basis matrix

Simplex tableau:



$$i\text{-th row: } x_{BC(i)} + \sum_{j \in N} \bar{a}_{ij} x_j = \bar{a}_{i0}$$

$$\text{with: } \bar{a}_{ij} := (B^{-1}A_j)_i, \quad \bar{a}_{i0} := (B^{-1}b)_i$$

N the set of indices of non-basic variables

Consider a row i where \bar{a}_{i0} is fractional.

$$\text{Then } x_{BC(i)}^* + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \stackrel{=0}{=} x_{BC(i)}^* = \bar{a}_{i0} > \lfloor \bar{a}_{i0} \rfloor$$

For all feasible integer solutions x :

$$x_{BC(i)} + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq x_{BC(i)} + \sum \bar{a}_{ij} x_j = \bar{a}_{i0}$$

$\uparrow \leq \bar{a}_{ij}$ $\nearrow \geq 0$

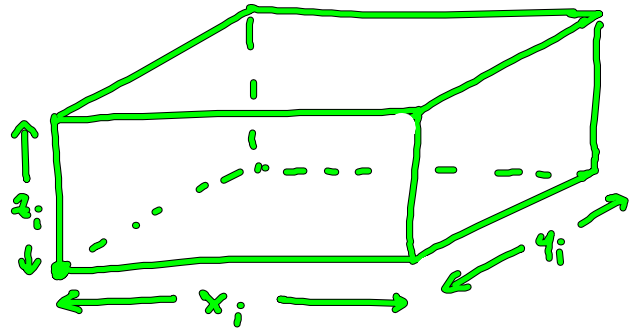
x int!
 \Rightarrow

$$x_{BC(i)} + \sum_{j \in N} \lfloor \bar{a}_{ij} \rfloor x_j \leq \lfloor \bar{a}_{i0} \rfloor$$

Gomory cut

Exercise 71.

Bricks given by their sizes



First ignore rotations.

► Variables: lower left front corner: $p_i^x, p_i^y, p_i^z \geq 0 \forall i$

total box size: s^x, s^y, s^z

→ objective function: $\min s^x + s^y + s^z$

some constraints:

$$\begin{aligned} p_i^x + x_i &\leq s^x \\ p_i^y + y_i &\leq s^y \\ p_i^z + z_i &\leq s^z \end{aligned}$$

► non-overlapping \Leftrightarrow given a pair of bricks, one must be completely to the left or to the right of the other w.r.t. at least one coordinate

$$\Leftrightarrow \forall (i,j): \begin{array}{ccccccc} p_i^x + x_i \leq p_j^x & \vee & p_j^x + x_j \leq p_i^x & \vee & & & \\ y & y & y & \vee & y & y & y \\ z & z & z & \vee & z & z & z \end{array}$$

→ introduce 6 0/1-variables $\delta_{(i,j)}^{x/y/z}$

e.g. $f_{(i,j)}^y = 0$ means that $p_j^y + y_j \leq p_i^y$ is satisfied

$$\updownarrow$$

$$p_i^y - p_j^y - y_j \geq 0$$

$f_{(i,j)}^y = 1$ means: a trivial constraint is satisfied:

$$p_i^y - p_j^y - y_j \geq -M \cdot f_{(i,j)}^y$$

with $M := \sum_{i=1}^n x_i + y_i + z_i$

→ non-overlapping constraints:

$$\forall (i,j): \quad p_j^x - p_i^x - x_i \geq -M \cdot f_{(i,j)}^x$$

$$p_j^y - p_i^y - y_i \geq -M \cdot f_{(i,j)}^y$$

$$p_j^z - p_i^z - z_i \geq -M \cdot f_{(i,j)}^z$$

$$f_{(i,j)}^x + f_{(i,j)}^y + f_{(i,j)}^z + f_{(j,i)}^x + f_{(j,i)}^y + f_{(j,i)}^z \leq 5$$

$$f_{(i,j)}^{x/y/z} \in \{0,1\}$$

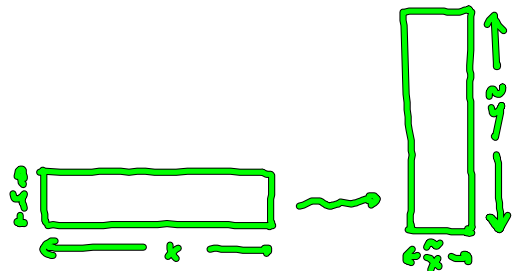
► rotations

(for 2 dimensions)

If the brick is rotated

then the sizes change:

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$$\begin{aligned}x &= y \\ \tilde{y} &= x\end{aligned}$$

→ decision variable $\sigma^{\text{rot}} \in \{0,1\}$
 $\sigma^{\text{nonrot}} \in \{0,1\}$

$$\begin{aligned}\rightarrow \tilde{x} &= \sigma^{\text{nonrot}} x + \sigma^{\text{rot}} y & \sigma^{\text{rot}} + \sigma^{\text{nonrot}} &= 1 \\ \tilde{y} &= \sigma^{\text{nonrot}} y + \sigma^{\text{rot}} x\end{aligned}$$

(in 3 dimensions)

$3! = 6$ different states corresponding to all permutations of the tuple (x, y, z)

→ 6 0/1-variables σ_i^π ($\pi \in S_3$) for every brick i
 plus a constraint $\sum_{\pi \in S_3} \sigma_i^\pi = 1$

$$\begin{aligned}\rightarrow \tilde{x}_i &= \sigma_i^{\text{id}} x_i + \sigma_i^{(x,y,z)} x_i + \dots \\ \tilde{y}_i &= \sigma_i^{\text{id}} y_i + \sigma_i^{(x,y,z)} z_i + \dots \\ \tilde{z}_i &= \sigma_i^{\text{id}} z_i + \sigma_i^{(x,y,z)} y_i + \dots\end{aligned}$$

→ use new sizes $\tilde{x}_i, \tilde{y}_i, \tilde{z}_i$ instead of x_i, y_i, z_i in all constraints