Chapter 1: Introduction

1.1 Variants of the linear programming problem

Example:

$$\text{minimize} \quad 2x_1 - x_2 + 4x_3$$

linear cost function
" objective "

$$\begin{aligned}
\text{subject to:} \quad x_1 + x_2 \qquad\quad + x_4 &\le 2 \\
3x_2 - x_3 \qquad\quad &= 5 \\
x_3 + x_4 &\ge 3 \\
x_1 \qquad\qquad &\ge 0 \\
x_3 \qquad &\le 0
\end{aligned}$$

linear equality and inequality constraints

special constraints

General linear program

Given: $c \in \mathbb{R}^n$, $a_i \in \mathbb{R}^n$ $i \in M_1 \dot\cup M_2 \dot\cup M_3$, $b_i \in \mathbb{R}$ $i \in M_1 \cup M_2 \cup M_3$

finite index sets

Task: Find $x \in \mathbb{R}^n$

$$\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & a_i^T \cdot x \ge b_i && \text{for } i \in M_1 \\
& a_i^T \cdot x = b_i && \text{for } i \in M_2 \\
& a_i^T \cdot x \le b_i && \text{for } i \in M_3 \\
& x_j \ge 0 && \text{for } j \in N_1 \\
& x_j \le 0 && \text{for } j \in N_2
\end{aligned} \quad (*)$$

x is called vector of (decision) variables.

$N_1, N_2 \subseteq \{1, \dots, n\}$

x satisfying $(*)$ is <u>feasible solution</u>

Feasible solution x^* with $c^T x^* \le c^T x$ $\forall$ feasible x is an <u>optimal solution</u>.

The problem is unbounded if $\forall k \in \mathbb{R}$ $\exists$ feasible x: $c^T x \le k$

Remark: $\max c^T x \iff \min (-c)^T \cdot x$

Remark: Any linear program (LP) can be written in the following form:

$$\begin{aligned}
\min \quad & c^T x \\
\text{s.t.} \quad & A \cdot x \ge b
\end{aligned}$$

$$A = \begin{pmatrix} - a_1 - \\ - a_2 - \\ \vdots \\ - a_m - \end{pmatrix} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

for some $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$

(Rewrite $a_i^T \cdot x = b_i$ as $a_i^T x \geq b_i \wedge -a_i^T \cdot x \geq -b_i$
$a_i^T \cdot x \leq b_i$ as $-a_i^T x \geq -b_i$)

Standard form problems

LP in standard form

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & A \cdot x = b \\ & x \geq 0 \end{aligned} \qquad c \in \mathbb{R}^n,\ A \in \mathbb{R}^{m \times n},\ b \in \mathbb{R}^m$$

Example: The "diet problem"

Given: n different foods, m different nutrients

a_{ij} = amount of nutrient i in one unit of food j

Let $b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ where b_i is requirement of nutrient i in some ideal diet.

Task: Find an ideal diet consisting of foods $1, \dots, n$

$$\begin{aligned} A \cdot x &= b \\ x &\geq 0 \end{aligned} \quad \text{where } x_j = \#\text{units of food } j$$

$$\begin{aligned} A \cdot x &\geq b \\ x &\geq 0 \end{aligned} \qquad \min c^T x$$

Reduction to standard form:

i) Elimination of free variables:

Replace x_j with $x_j^+, x_j^- \geq 0$: $x_j := x_j^+ - x_j^-$

ii) Elimination of inequality constraints:

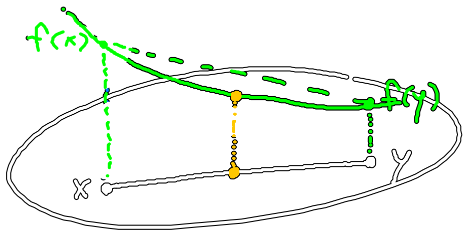
$$a^T x \leq b \quad \longrightarrow \quad \begin{aligned} a^T x + s &= b \\ s &\geq 0 \end{aligned} \quad \text{"slack variable"}$$

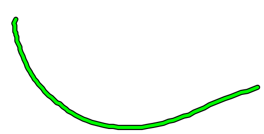
$$\bar{a}^T x \geq \bar{b} \quad \longrightarrow \quad \begin{aligned} \bar{a}^T x - \bar{s} &= \bar{b} \\ \bar{s} &\geq 0 \end{aligned}$$

1.3 Piecewise linear convex objective functions

Def: i) $f: \mathbb{R}^n \to \mathbb{R}$ is called $\overset{\text{concave}}{\text{convex}}$ if

$$f(\lambda \cdot x + (1-\lambda) \cdot y) \overset{\geq}{\leq} \lambda \cdot f(x) + (1-\lambda) \cdot f(y) \quad \forall x, y \in \mathbb{R}^n,\ \lambda \in [0,1]$$

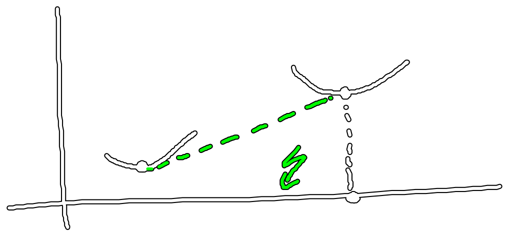




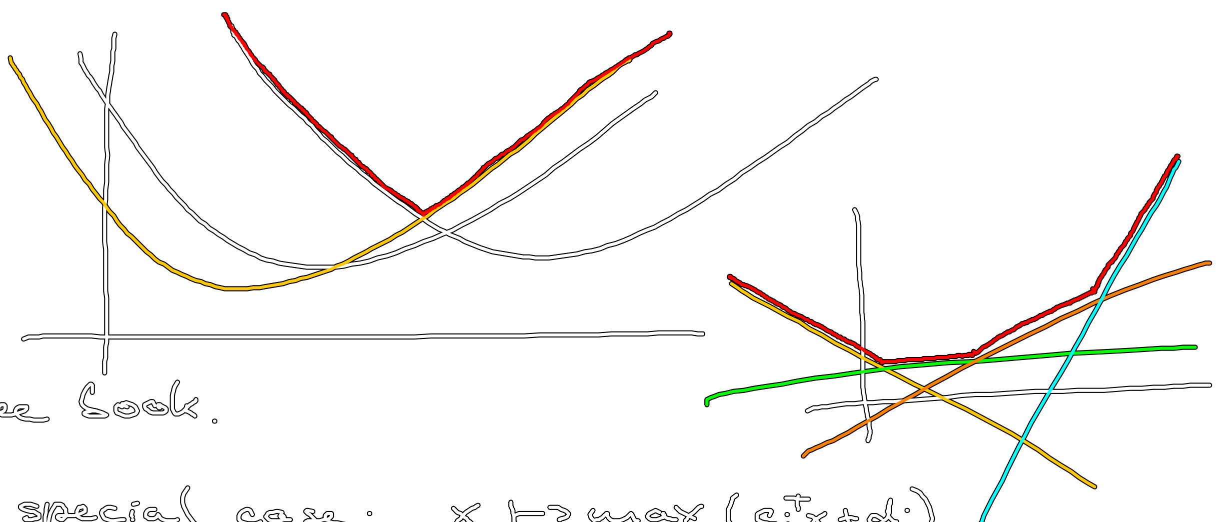
Notice: f convex $\Leftrightarrow$ $-f$ concave.

Example: $f(x) = a_0 + \sum_{i=1}^{n} a_i \cdot x_i$ (affine linear) is both convex and concave

Remark: Convex functions play an important role in optimization because every local minimum is a global minimum.



Theorem: If $f_1, \dots, f_k : \mathbb{R}^n \to \mathbb{R}$ are convex, then $f : \mathbb{R}^n \to \mathbb{R}$ defined by $f(x) := \max_{i=1,\dots,k} f_i(x)$ is also convex

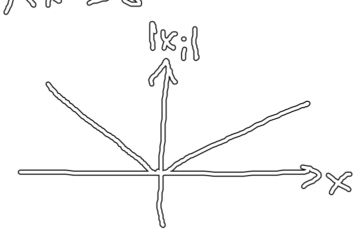


Proof: see book.

Important special case: $x \mapsto \max_{i=1,\dots,m} (c_i^T x + d_i)$

$$\begin{array}{ll} \min & \max_{i=1,\dots,m} (c_i^T x + d_i) \\ \text{s.t.} & Ax \geq b \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & z \\ \text{s.t.} & z \geq c_i^T x + d_i \quad \forall i = 1,\dots,m \\ & Ax \geq b \end{array}$$

Example: $c_1, \dots, c_n \geq 0$



$$\begin{array}{ll} \min & \sum_{i=1}^{n} c_i \cdot |x_i| \\ \text{s.t.} & Ax \geq b \end{array} \quad \Leftrightarrow \quad \begin{array}{ll} \min & \sum_{i=1}^{n} c_i \cdot z_i \\ \text{s.t.} & x_i \leq z_i \\ & -x_i \leq z_i \\ & A \cdot x \geq b \end{array}$$

$$\Leftrightarrow \quad \begin{array}{ll} \min & \sum_{i=1}^{n} c_i \cdot (x_i^+ + x_i^-) \\ \text{s.t.} & A \cdot x^+ - A \cdot x^- \geq b \end{array}$$

1.4 Graphical representation and solution

Example:

$$c = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\min \ -x_1 - x_2$$

s.t. $\boxed{x_1 + 2x_2 \leq 3}$

$\boxed{2x_1 + x_2 \leq 3}$

$\boxed{x_1, x_2 \geq 0}$