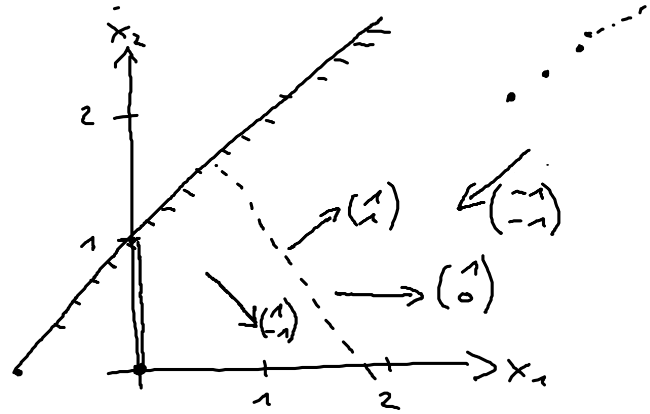
Example: $c = \binom{c_1}{c_2}$

$$\begin{aligned} \min\ & c_1 x_1 + c_2 x_2 \\ \text{s.t. } & \boxed{-x_1 + x_2 \leq 1} \\ & x_1 \geq 0 \\ & x_2 \geq 0 \\ & x_2 \leq -1 \end{aligned}$$



4 cases can occur:

i) there exists a unique opt. sol.

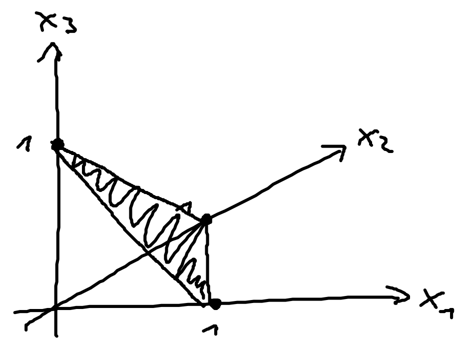
ii) there exist infinitely many opt. sol. and the set of opt. sol. is either bounded or unbounded

iii) opt. sol. value is $-\infty$, no feasible solution is optimal

iv) feasible set is empty

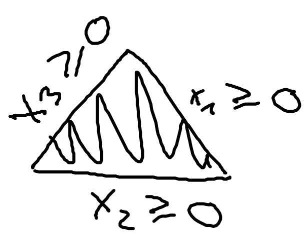
Visualizing problems in standard form

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0 \end{aligned}$$



Example:

$$x_1 + x_2 + x_3 = 1$$
$$x_1, x_2, x_3 \geq 0$$



Chapter 2: The geometry of linear progr.

2.1 Polyhedra and convex sets

Def: Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$. Then the set

$$\{x \in \mathbb{R}^n \mid A \cdot x \geq b\}$$

is a <u>polyhedron</u>.

In particular, the set of feasible solutions to an LP is a polyhedron.

A set of the form $\{x \in \mathbb{R}^n \mid A \cdot x = b, x \geq 0\}$ is a <u>polyhedron in standard form</u>.

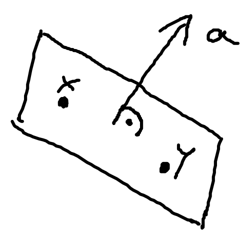
<u>Def:</u> A set $S \subseteq \mathbb{R}^n$ is <u>bounded</u> if there is a constant $K \in \mathbb{R}$ such that

$$\|x\|_\infty \leq K \quad \forall x \in S.$$

<u>Def:</u> Let $a \in \mathbb{R}^n \setminus \{0\}$ and $b \in \mathbb{R}$

(i) The set $\{x \in \mathbb{R}^n \mid a^T \cdot x = b\}$ is a <u>hyperplane</u>

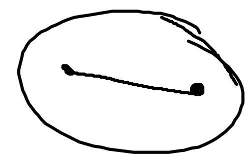
(ii) The set $\{x \in \mathbb{R}^n \mid a^T x \geq b\}$ is a <u>halfspace</u>



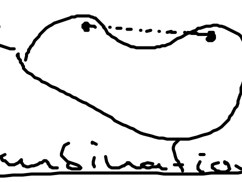
$$a^T x = a^T \cdot y = b$$
$$\Rightarrow a^T \cdot (x - y) = 0$$

<u>Def:</u> $S \subseteq \mathbb{R}^n$ is convex, if

$$\lambda \cdot x + (1-\lambda) \cdot y \in S \quad \forall x, y \in S, \lambda \in [0,1]$$

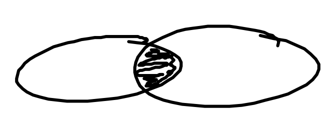


<u>Def:</u> Let $x^1, \ldots, x^k \in \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ with $\lambda_1, \ldots, \lambda_k \geq 0$ and $\lambda_1 + \ldots + \lambda_k = 1$



(i) The vector $\sum_{i=1}^{k} \lambda_i \cdot x^i$ is a <u>convex combination</u> of $x^1, \ldots, x^k$.

(ii) The <u>convex hull</u> of $x^1, \ldots, x^k$ is the set of all convex combinations of those vectors.



<u>Theorem:</u>

(i) The intersection of convex sets is convex

(ii) Every polyhedron is convex.

(iii) A convex combination of a finite number of elements of a convex set also belongs to that set.

(iv) The convex hull of a finite number of vectors is a convex set.

Proof: (i) For $i \in I$ let S_i be a convex set

let $x, y \in \bigcap_{i \in I} S_i$ and $\lambda \in [0,1]$

S_i is convex $\Rightarrow \lambda \cdot x + (1-\lambda) \cdot y \in S_i \quad \forall i \in I$

$\Rightarrow \lambda \cdot x + (1-\lambda) \cdot y \in \bigcap_{i \in I} S_i$

(ii) $P = \{ x \mid A \cdot x \geq b \}$

$x, y \in P$, $\lambda \in [0,1]$

$A \cdot (\lambda \cdot x + (1-\lambda) \cdot y) = \lambda \cdot \underbrace{Ax}_{\geq b} + (1-\lambda) \cdot \underbrace{Ay}_{\geq b} \geq b$

(iii) Proof by induction over # elements k.

$k = 2$ by def.

$k \to k+1 \quad x^1, \dots, x^{k+1} \in S, \lambda_1 \dots \lambda_{k+1} \geq 0 \ \sum_{i=1}^{k+1} \lambda_i = 1$

$\sum_{i=1}^{k+1} \lambda_i \cdot x^i$ Assume w.l.o.g. $\lambda_{k+1} < 1$

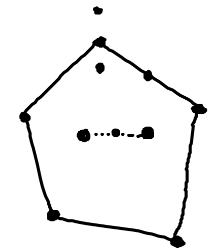
$= \lambda_{k+1} \cdot x^{k+1} + (1-\lambda_{k+1}) \cdot \underbrace{\sum_{i=1}^{k} \frac{\lambda_i}{(1-\lambda_{k+1})} \cdot x^i}_{\substack{\text{convex comb.} \\ \text{of } x^1, \dots, x^k \\ \in S}} \in S$

(iv) $x = \sum_{i=1}^{k} \lambda_i \cdot x^i$, $y = \sum_{i=1}^{k} \mu_i x^i \quad \lambda_i, \mu_i \geq 0 \ \sum_i \lambda_i = \sum_i \mu_i = 1$

$\lambda \cdot x + (1-\lambda) \cdot y = \sum_{i=1}^{k} (\lambda \cdot \lambda_i + (1-\lambda) \cdot \mu_i) \cdot x^i$

$\sum_{i=1}^{k} (\lambda \cdot \lambda_i + (1-\lambda) \cdot \mu_i) = \lambda \cdot \underbrace{\sum_{i=1}^{k} \lambda_i}_{=1} + (1-\lambda) \cdot \underbrace{\sum_{i=1}^{k} \mu_i}_{=1} = 1 \quad \square$

2.2. Extreme points, vertices, and basic feasible sol.



Def: Let P be a polyhedron in $\mathbb{R}^n$

(i) $x \in P$ is an *extreme point* of P if

$$x \neq \lambda \cdot y + (1-\lambda)\cdot z \quad \forall y, z \in P \setminus \{x\},\ \lambda \in [0,1]$$

(ii) $x \in P$ is a *vertex* of P if there exists $c \in \mathbb{R}^n$ such that $c^T x < c^T y \ \forall y \in P \setminus \{x\}$.

Def: Let $P \subseteq \mathbb{R}^n$ be a polyhedron defined by

$$a_i^T x \geq b_i \quad \text{for } i \in M_1$$
$$a_i^T \cdot x = b_i \quad \text{for } i \in M_2$$
$$a_i^T \cdot x \leq b_i \quad \text{for } i \in M_3$$

If a vector x^* satisfies $a_i^T \cdot x^* = b_i$ for some i then the corresp. constraint is *active* or *binding* at x^*

Theorem: Let $x^* \in \mathbb{R}^n$ and $I = \{i \mid a_i^T \cdot x^* = b_i\}$. The following are equivalent:

(i) There exist n vectors in $\{a_i \mid i \in I\}$ which are linearly independent.

(ii) The vectors in $\{a_i \mid i \in I\}$ span $\mathbb{R}^n$.

(iii) The system of equations $a_i^T \cdot x = b_i$ for $i \in I$ has a unique solution.