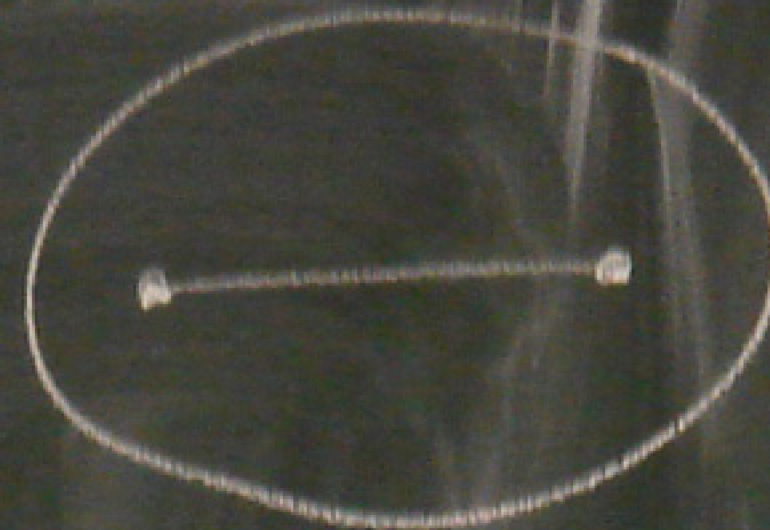
$\min \; c^T x$

$\text{s.t.} \; Ax \geq b$



$P = \{x \in \mathbb{R}^n \mid Ax \geq b\}$ polyhedron

$\{x \mid a^T x = b\}$ Hyperplane for $a \neq 0$

$\{x \mid a^T x \geq b\}$ Halfspace

$x^1, \ldots, x^k \in \mathbb{R}^n$, $\lambda_1, \ldots, \lambda_k \geq 0$, $\sum \lambda_i = 1$

$\sum_{i=1}^{k} \lambda_i x^i$ convex combination.

Def.: (i) x extreme point if

$$x \neq \lambda \cdot y + (1-\lambda) z \quad \forall y, z \in P \setminus \{x\},\ \lambda \in [0,1]$$

(ii) x is vertex if $\exists\, c \in \mathbb{R}^n$ such that

$$c^T x < c^T y \quad \forall y \in P \setminus \{x\}$$

$a_i^T x \geq b_i$ for $i \in M_1$

$a_i^T x = b_i$ for $i \in M_2$

$\cancel{(a_i^T x \leq b_i \text{ for } i \in M_3)}$

If x^* satisfies $a_i^T x^* = b_i$ for some i then the corresp. constr. is binding or active

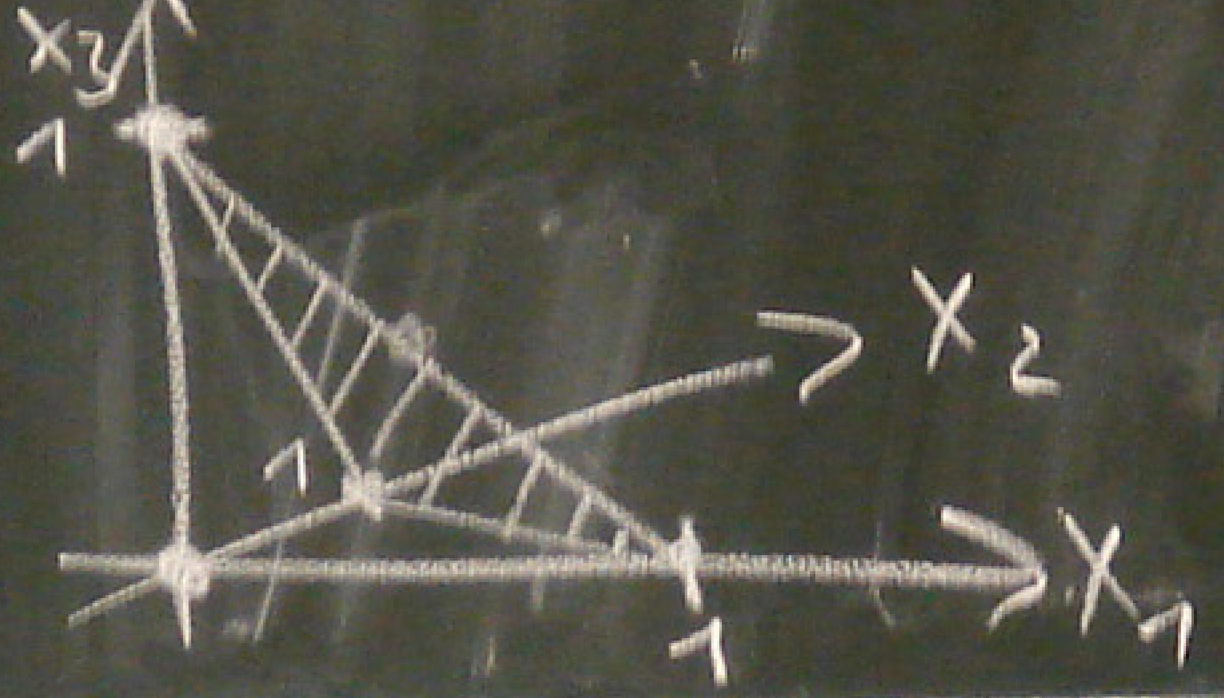
Def: (i) The vector $x^* \in \mathbb{R}^n$ is a basic solution if

a) All equality constraints are active

b) There are n linearly independent constraints that are act.

(ii) A basic sol. satisfying all constr. is a basic feasible sol.

Example: $P = \{x \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 1,\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\}$



Theorem: For $x^* \in P$ (polyhedron, $P = \{x \in \mathbb{R}^n \mid a_i^T x \geq b_i \;\; i \in M_1,\; a_i^T x = b_i \;\; i \in M_2\}$)

the following are equivalent:

(i) x^* vertex (ii) x^* extreme point (iii) x^* basic feasible sol.

Proof: (i) $\Rightarrow$ (ii) $c^T x^* < c^T y \;\; \forall y \in P \setminus \{x^*\}$

Assume by contradiction that $\exists y, z \in P \setminus \{x^*\}$, $\lambda \in [0,1]$

$$x^* = \lambda \cdot y + (1-\lambda) \cdot z$$

$$c^T x^* = \lambda \cdot \underbrace{c^T y}_{< c^T x^*} + (1-\lambda) \cdot \underbrace{c^T z}_{< c^T x^*} < \lambda \cdot c^T x^* + (1-\lambda) \cdot c^T x^* = c^T x^* \;\; \lightning$$

(ii) $\Rightarrow$ (iii) Assume that x^* is not a basic feas. sol.

$$I = \{i \mid a_i^T x^* = b_i\} \Rightarrow \exists d \in \mathbb{R}^n \setminus \{0\} : a_i^T d = 0 \;\; \forall i \in I$$

For $\varepsilon > 0$ consider $y := x^* + \varepsilon \cdot \underline{d}$

$$z := x^* - \varepsilon \cdot \underline{d}$$

$$\Longrightarrow a_i^T y = a_i^T x^* = a_i^T z = \delta_i \quad \text{for } i \in \underline{I}$$

For $i \notin \underline{I}$: $a_i^T x^* > \delta_i \Longrightarrow \begin{matrix} a_i^T y > \delta_i \\ a_i^T z > \delta_i \end{matrix}$ for ε small.

$$\Longrightarrow y, z \in P \setminus \{x^*\} \text{ and } x^* = \tfrac{1}{2} \cdot y + \tfrac{1}{2} \cdot z \quad \lightning$$

$\underline{(iii) \Rightarrow (i):}$ $I = \{ i \mid a_i^T x^* = b_i \}$

$$c = \sum_{i \in I} a_i \qquad c^T x^* = \sum_{i \in I} a_i^T x^* = \sum_{i \in I} b_i$$

For $x \in P$: $a_i^T x \geq b_i \ \forall i \in I$

$$\Rightarrow c^T x = \sum_{i \in I} a_i^T x \geq \sum_{i \in I} b_i$$

For $x \neq x^*$ $\exists i_0 \in I$: $a_{i_0}^T x > b_{i_0}$

$$\Rightarrow c^T x > \sum_{i \in I} b_i \qquad \square$$

Corollary: The number of vertices and basic solutions of a polyhedron is finite ($\leq \binom{m}{n}$ for m inequality constr.)

Example: $P = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, \; i = 1, \ldots, n\}$ unit cube

$\#$ vertices $= 2^n$, $\#$ constraints $= 2n$

Def.: Two distinct basic solutions are <u>adjacent</u> if there are $n-1$ linearly indep. constraints that are active at both of them. If both solutions are feasible then the line segment that joins them is an <u>edge</u> of the polyhedron

2.3 Polyhedra in standard form

Let $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$, $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$

Observation: We can assume w.l.o.g. that $\operatorname{rank}(A) = m$.

Because: Assume $a_i = \sum_{j \neq i} \lambda_{ij} a_j$

Case 1: $b_i = \sum_{j \neq i} \lambda_{ij} b_j$

Then $a_j^T x = b_j$ for $j \neq i$ implies $a_i^T x = \sum_{j \neq i} \lambda_{ij} a_j^T x = \sum_{j \neq i} \lambda_{ij} b_j = b_i$ and the ith constraint is redundant and can be deleted.

Case 2: $b_i \neq \sum_{j \neq i} \lambda_{ij} b_j \Rightarrow Ax = b$ has no solution. $\square$