

$$P = \{x \in \mathbb{R}^n \mid A \cdot x = \delta, x \geq 0\} \quad A \in \mathbb{R}^{m \times n}, \text{rank}(A) = m$$

basic solution can be specified by choosing  $m$  linearly indep. columns of  $A$ :  $(A_{B(1)}, \dots, A_{B(m)}) =: B$

$$x_B = \begin{pmatrix} x_{B(1)} \\ \vdots \\ x_{B(m)} \end{pmatrix} = B^{-1} \cdot \delta, \quad x_j = 0 \text{ for } j \notin B(1), \dots, B(m)$$

$$A \cdot x = B \cdot x_B + \sum_{j \notin B(1), \dots, B(m)} A_j \cdot x_j = B \cdot B^{-1} \cdot \delta + 0 = \delta$$

$$\sum_{j=1}^n A_j \cdot x_j$$

## 2.5 Existence of extreme points

Def: A polyhedron  $P$  contains a line if there is a  $x \in P$  and some direction  $d \neq 0$  such that  $x + \lambda \cdot d \in P \forall \lambda \in \mathbb{R}$ .

Theorem: Let  $P = \{x \in \mathbb{R}^n \mid Ax \geq \delta\} \neq \emptyset$ . The following are equivalent:

- (i) There is an extreme point  $x \in P$
- (ii)  $P$  does not contain a line
- (iii) matrix  $A$  contains  $n$  linearly independent rows.

Proof: (i)  $\Rightarrow$  (iii): Clear by def. of basic feasible solutions.

(iii)  $\Rightarrow$  (ii): Assume by contradiction that  $x \in P$  and

$$x + \lambda \cdot d \in P \quad \forall \lambda \in \mathbb{R} \quad \text{for some } d$$

$$\Rightarrow A \cdot (x + \lambda \cdot d) = A \cdot x + \lambda \cdot A \cdot d \geq \delta \quad \forall \lambda \in \mathbb{R}$$

$$\Rightarrow A \cdot d = 0$$

$$\text{Since } \text{rank}(A) = n \Rightarrow d = 0$$

(ii)  $\Rightarrow$  (i): Let  $x \in P$  such that  $|\{i \mid a_i^T \cdot x = \delta_i\}|$  is maximum. Assume by contradiction that  $\{a_i \mid a_i^T \cdot x = \delta_i\}$  does not contain  $n$  linearly indep. vectors.

$$\Rightarrow \exists d \in \mathbb{R}^n \setminus \{0\} \text{ such that } a_i^T \cdot d = 0 \quad \forall i \in I$$

$$\Rightarrow a_i^T \cdot (x + \lambda \cdot d) = \delta_i \quad \forall \lambda \in \mathbb{R}$$

Since  $x + \lambda \cdot d$  is not contained in  $P$  for all  $\lambda \in \mathbb{R}$ , there is a constraint  $a_j^T x \geq \delta_j$  that is eventually violated for some  $\lambda \in \mathbb{R}$ .  $\rightarrow \exists \lambda_0 \in \mathbb{R}$  with

$$a_j^T (x + \lambda_0 \cdot d) = \delta_j \text{ and } x + \lambda_0 \cdot d \in P.$$

This contradicts the choice of  $x$ .  $\square$

Corollary: Every non-empty bounded polyhedron and every non-empty polyhedron in standard form contain an extreme points.

$$\begin{aligned} Ax = \delta \\ I \cdot x \geq 0 \end{aligned} \iff \begin{pmatrix} A \\ -A \\ I \end{pmatrix} \cdot x \geq \begin{pmatrix} \delta \\ -\delta \\ 0 \end{pmatrix}$$

Example:

$$\left. \begin{aligned} x_1 + x_2 &\geq 1 \\ x_1 + 2x_2 &\geq 0 \end{aligned} \right\} P$$

$x_1, x_2, x_3$   
contains a line  
because

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} \in P \quad \forall \lambda \in \mathbb{R}$$

## 2.6 Optimality of extreme points

Theorem: Let  $P \subseteq \mathbb{R}^n$  polyhedron and  $c \in \mathbb{R}^n$ . If  $P$  has an extreme point and  $c^T x$  is bounded from below for  $x \in P$ , then there exists an extreme point that is optimal.

Proof: We prove that for any  $x \in P$  there exists an extreme point  $y \in P$  with  $c^T y \leq c^T x$ .

Let  $x \in P$  and  $I := \{i \mid a_i^T x = \delta_i\}$ . If  $\{a_i \mid i \in I\}$  contains a linearly indep. vectors we are done. Otherwise  $\exists d \in \mathbb{R}^n$ ,  $d \neq 0$  with  $a_i^T \cdot d = 0 \quad \forall i \in I$ . Assume that  $c^T \cdot d \leq 0$  (otherwise replace  $d$  by  $-d$ ).

1. Case:  $c^T \cdot d < 0 \Rightarrow c^T \cdot (x + \lambda \cdot d) \rightarrow -\infty$  for  $\lambda \rightarrow \infty$

Thus  $\exists j : a_j^T \cdot d < 0$ . Let  $\lambda_0 := \min_{j \notin I : a_j^T d < 0} \frac{\delta_j - a_j^T x}{a_j^T d}$

$$\text{Remark: } a_j^T (x + \lambda_0 \cdot d) = a_j^T x + \lambda_0 \cdot a_j^T d \geq \delta_j$$

Then  $x + \lambda_0 \cdot d \in P$  and  $\{i \mid a_i^T(x + \lambda_0 \cdot d) = b_i\} \supseteq \{i \mid a_i^T x = b_i\}$  contains more linearly independent vectors than  $\{a_i \mid i \in I\}$ .

Notice that  $c^T x \geq c^T(x + \lambda_0 \cdot d)$ . Repeat argument with  $x + \lambda_0 \cdot d$ .

2. Case:  $c^T \cdot d = 0 \Rightarrow c^T(x + \lambda \cdot d) = c^T x \quad \forall \lambda \in \mathbb{R}$

Since  $P$  does not contain a line, we find an additional constraint in direction  $d$  or  $-d$ .

See above ... □

Corollary: For any linear programming problem over a nonempty polyhedron, either the optimal cost is  $-\infty$  or there exists an optimal solution.

Proof: Any linear program is equivalent to a linear program in standard form.

Example:  $\min \frac{1}{x}$  does not have opt. sol.  
s.t.  $x > 0$

### Chapter 3: The Simplex Method

Let  $A \in \mathbb{R}^{m \times n}$ ,  $b \in \mathbb{R}^m$ ,  $c \in \mathbb{R}^n$ ,  $\text{rank}(A) = m$

Let  $P = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$  and consider the LP

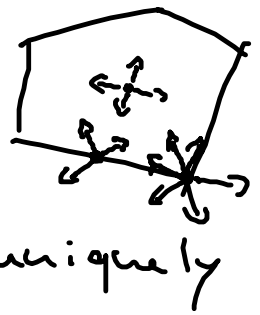
$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

#### 3.1 Optimality conditions

Def: For  $x \in P \subseteq \mathbb{R}^n$  (polyhedron), the vector  $d \in \mathbb{R}^n \setminus \{0\}$  is a feasible direction at  $x$  if there exists  $\theta > 0$

with  $x \in \Theta \cdot d \in P$ .

Observation: Let  $B = (A_{B(1)}, \dots, A_{B(m)})$  be a basis matrix. Then the basic variables  $x_{B(1)}, \dots, x_{B(m)}$  in the system  $Ax = b$  are uniquely determined by the non-basic variables.



$$Ax = b \Leftrightarrow B \cdot x_B + \sum_{j \notin B(1), \dots, B(m)} A_j \cdot x_j = b$$

$$\Leftrightarrow x_B = B^{-1} \cdot b - \sum_{j \notin \dots} B^{-1} \cdot A_j \cdot x_j$$