3.1. Optimality conditions

$$\min c^T x \quad \text{s.t. } Ax = b,\ x \geq 0 \qquad A \in \mathbb{R}^{m \times n},\ \operatorname{rank}(A) = m$$

$\underbrace{(A_{B(1)}, \ldots, A_{B(m)})}_{B}$ basis

$$Ax = b \iff B \cdot x_B + \sum_{j \neq B(1), \ldots, B(m)} A_j \cdot x_j = b$$

$$\iff x_B = B^{-1} \cdot b - \sum_{j \neq \ldots} B^{-1} \cdot A_j \cdot x_j$$

Consider a basic feasible solution x given by $x_B = B^{-1} \cdot b \geq 0$

For some fixed $j \neq B(1), \ldots, B(m)$, let $d \in \mathbb{R}^n$ be given by $d_B := -B^{-1} \cdot A_j$, $d_j := 1$ and $d_{j'} := 0$ for all $j' \neq j, B(1), \ldots, B(m)$

Then $A \cdot (x + \theta d) = b \quad \forall \theta \in \mathbb{R}$

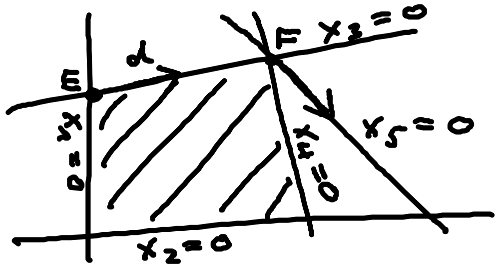
and we call d the j-th basic direction.

Question: Is d a feasible direction at x?

Case 1: If x is a non-degenerate basic feasible solution, then $x_B > 0$ and $x + \theta \cdot d \geq 0$ for $\theta > 0$ small enough. $\Rightarrow$ Answer is yes.

Case 2: If x is degenerate then d is not necessarily a feasible direction. E.g. if $x_{B(i)} = 0$ and $d_{B(i)} < 0$, then $x + \theta \cdot d \ngeq 0$ for $\theta > 0$.

Example: $n = 5$, $m = 3$, $n - m = 2$



in F: basic variables are x_1, x_2, x_4

How does the cost change when moving along a basic direction?

$$c^T(x+\theta\cdot d) = c^T\cdot x + \theta\cdot c^T\cdot d = c^T x + \theta\cdot\underbrace{(c_j - c_B^T\cdot B^{-1}\cdot A_j)}_{\bar{c}_j :=}$$

Def: For $j=1,\dots,n$ the reduced cost of variable x_j is $\bar{c}_j := c_j - c_B^T\cdot B^{-1}\cdot A_j$.

$B^{-1}\cdot B = \begin{pmatrix}1 & & \\ & \ddots & \\ & & 1\end{pmatrix}$

Notice that $\bar{c}_{B(i)} = c_{B(i)} - c_B^T\cdot \underbrace{B^{-1}\cdot A_{B(i)}}_{e_i}$

$$= c_{B(i)} - c_{B(i)} = 0$$

Theorem: Let x be a basic feasible solution and $\bar{c}$ the vector of reduced costs.

(i) If $\bar{c}\geq 0$, then x is optimal

(ii) If x is optimal and non-degenerate, then $\bar{c}\geq 0$.

Proof: (i) Let $y\in P$, then

$$c^T y = c_B^T\cdot y_B + \sum_{j\neq B(1),\dots,B(m)} c_j\cdot y_j$$

$$= c_B^T\cdot\Big(B^{-1}\cdot b - \sum_{j\neq\dots\dots} B^{-1}\cdot A_j\, y_j\Big) + \sum_{j\neq\dots\dots} c_j y_j$$

$$= c_B^T\cdot(B^{-1}\cdot b) + \sum_{j\neq\dots\dots}\underbrace{(c_j - c_B^T\cdot B^{-1}\cdot A_j)}_{\bar{c}_j}\, y_j$$

$$\geq c_B^T\cdot(B^{-1}\cdot b)$$

$$= c_B^T\cdot x_B = c^T x.$$

(ii) Assume by contradiction $\bar{c}_j < 0$ for some $j\neq B(1),\dots,B(m)$. Since x is a non-deg. basic feasible solution, the j-th basic direction d is a feasible direction with $\underbrace{c^T\cdot d}_{\bar{c}_j} < 0 \Rightarrow x$ not optimal ↯ □

Def: A basis matrix B is <u>optimal</u> if

(i) $B^{-1}\cdot b\geq 0$ and

(ii) $\bar{c}^T = c^T - c_B^T \cdot B^{-1} \cdot A \geq 0$.

In particular, the associated basic solution is feasible (by (i)) and optimal (by (ii)).

3.2 Development of the simplex method

Assumption for now: Only non-degenerate basic feasible solutions.

Let x be a basic feasible solution and $\bar{c}_j < 0$ for some $j \neq B(1), \dots, B(m)$ (otherwise x is optimal)

Let d be the j-th basic direction. Then $\bar{c}_j = c^T \cdot d < 0$ and it is desirable to go to $x + \theta^* \cdot d$ with $\theta^* = \max\{\theta \geq 0 \mid x + \theta \cdot d \in P\}$

By construction of d, $A(x + \theta \cdot d) = b \ \forall \theta \in \mathbb{R}$

$$\Rightarrow \quad (x + \theta \cdot d \in P \iff x + \theta \cdot d \geq 0)$$

Case 1: $d \geq 0 \Rightarrow x + \theta \cdot d \geq 0 \ \forall \theta \geq 0 \Rightarrow \theta^* = \infty$

$\Rightarrow$ LP unbounded.

Case 2: $d_j < 0$ for some j: $x_j + \theta d_j \geq 0 \iff \theta \leq \frac{-x_j}{d_j}$

$$\Rightarrow \theta^* := \min_{j : d_j < 0} -\frac{x_j}{d_j} = \min_{\substack{i = 1, \dots, m \\ d_{B(i)} < 0}} -\frac{x_{B(i)}}{d_{B(i)}} > 0$$

Let $y := x + \theta^* \cdot d$ and $l \in \{1, \dots, m\}$ with $\theta^* = -\frac{x_{B(l)}}{d_{B(l)}}$

$\Rightarrow y_j = \theta^*, \ y_{B(l)} = 0$

Thus, x_j should replace $x_{B(l)}$ as a basic variable.

$\longrightarrow$ new basis matrix

$$\bar{B} = \left(A_{B(1)} \dots A_{B(l-1)} \, A_j \, A_{B(l+1)} \dots A_{B(m)} \right)$$

$$= \left(A_{\bar{B}(1)} \cdots A_{\bar{B}(m)} \right)$$

with

$$\bar{B}(i) = \begin{cases} B(i) & \text{if } i \neq \ell \\ j & \text{if } i = \ell \end{cases}$$

<u>Theorem</u>: Assume that $\Theta^* < \infty$

(i) $A_{\bar{B}(1)} \cdots A_{\bar{B}(m)}$ are linearly independent and $\bar{B}$ is a basis matrix

(ii) $y = x + \Theta^* \cdot d$ is a basic feasible sol. associated with $\bar{B}$.

<u>Proof</u>: $B^{-1} \cdot \bar{B} = (e_1, e_2, \dots e_{\ell-1} \; -d_B \; e_{\ell+1} \dots e_m)$

since $d_{B(\ell)} < 0$, the columns of $B^{-1} \cdot \bar{B}$ are linearly indep. $\Rightarrow$ columns of $\bar{B}$ are lin. indep.