

LP in standard form

$$\begin{array}{lll} \min & c^T x & A \in \mathbb{R}^{m \times n} \quad \text{rank}(A) = m \\ \text{s.t.} & Ax = b & c \in \mathbb{R}^n \\ & x \geq 0 & b \in \mathbb{R}^m \quad x \in \mathbb{R}^n \end{array}$$

$B =$
 $(A_{B(1)} \dots A_{B(m)})$ basis

\rightarrow basic solution $x_B := B^{-1} \cdot b$, $x_j = 0$ for $j \neq B(1), \dots, B(m)$
 ≥ 0 (feasibility)

j -th basic direction: increase non-basic variable x_j
and adjust the values of $x_{B(1)}, \dots, x_{B(m)}$

\rightarrow vector d with $d_B = -B^{-1} \cdot A_j$

change of cost: $c^T \cdot d = \bar{c}_j = c_j - c_B^T \cdot B^{-1} \cdot A_j$

How far can we walk in direction d ?

$$x \rightarrow x + \Theta \cdot d \quad \Theta \geq 0$$

What is the maximum Θ that can be chosen:

$$\Theta^* := \min_{j: d_j < 0} -\frac{x_j}{d_j} = \min_{i: d_{B(i)} < 0} -\frac{x_{B(i)}}{d_{B(i)}} = -\frac{x_{B(r)}}{d_{B(r)}}$$

\rightarrow new feasible solution $y := x + \Theta^* \cdot d$ with $y_{B(r)} = 0$
and $y_j := \Theta^*$ and $y_i = 0$ for all $i \neq B(1), \dots, B(m), j$

\rightarrow new basis $(A_{B(1)} \dots A_{B(r-1)} A_j A_{B(r+1)} \dots A_{B(m)}) = \bar{B}$

$$B^{-1} \cdot \left(\begin{array}{c} \vdots \\ 1 \\ \vdots \\ d_{B(r)} \\ \vdots \\ 1 \\ \vdots \end{array} \right) = \left(\begin{array}{c} \vdots \\ 1 \\ \vdots \\ d_{B(r)} \\ \vdots \\ 1 \\ \vdots \end{array} \right)$$

y is the basic feasible solution corresponding to
the new basis \bar{B}

Proof: By construction, $y_j = 0$ for $j \neq \bar{B}(1), \dots, \bar{B}(m)$
 and $A \cdot y = b \rightarrow y_{\bar{B}} = \bar{B}^{-1} \cdot b$. \square

Notice that $c^T y = c^T x + \theta^* \cdot c^T d = c^T x + \underbrace{\theta^*}_{>0} \cdot \underbrace{\bar{c}_j}_{<0} < c^T x$.

$Ax = b \leftarrow m$ active constraints
 $x \geq 0 \leftarrow m-m$ active non-negativity constraints

Example: $n = 6, m = 4 \rightarrow n - m = 2$

Initial basic feasible solution. Basic variables are x_1, x_2, x_3, x_6

Basic direction corresponding to x_4 is g .
 " " " " x_5 is f .

Choosing x_4 to enter the basis, x_6 must leave the basis. \rightarrow new basis with basic variables x_1, x_2, x_3, x_4

Basic direction corresponding to x_6 is $-g$
 " " " " x_5 is h .

The reduced cost of x_6 is $\bar{c}_6 > 0$
 " " " " x_5 is $\bar{c}_5 < 0$!

