Linear and Integer Programming (ADM II)

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An iteration of the simplex method

Let $B = (A_{B(1)} \dots A_{B(m)})$ be a basis matrix with a corresponding basic feasible solution x.

1. Let $\overline{c}^T := c^T - c_B{}^T B^{-1} A$. If $\overline{c} \geq 0$, then STOP; else choose j with $\overline{c}_j < 0$.
2. Let $u := B^{-1} A_j$. If $u \leq 0$, then STOP (optimal cost is $-\infty$).
3. Let $\theta^* := \min_{i: u_i > 0} \frac{x_{B(i)}}{u_i} = \frac{x_{B(\ell)}}{u_\ell}$ for some $\ell \in \{1, \dots, m\}$.
4. Form new basis by replacing $A_{B(\ell)}$ with A_j; corresponding basic feasible solution y is given by

$$y_j := \theta^* \qquad \text{and} \qquad y_{B(i)} = x_{B(i)} - \theta^* u_i \quad \text{for } i \neq \ell.$$

Remark

We say that the nonbasic variable x_j *enters the basis* and the basic variable $x_{B(\ell)}$ *leaves the basis*.

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Correctness of the simplex method

Theorem

Assume that the feasible set is nonempty and that every basic feasible solution is nondegenerate. Then, the simplex method terminates after a finite number of iterations. At termination, there are the following two possibilities:

1. *We have an optimal basis matrix B and an associated basic feasible solution x which is optimal.*
2. *We have found a vector d satisfying $Ad = 0$, $d \geq 0$, and $c^T d < 0$, and the optimal cost is $-\infty$.*

Prof sketch.

The simplex method makes progress in every iteration. Since there are only finitely many different basic feasible solutions, it stops after a finite number of iteration. □

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The simplex method for degenerate problems

- An iteration of the simplex method can also be applied if x is a degenerate basic feasible solution.
- In this case it might happen that $\theta^* := \min_{i:u_i>0} \frac{x_{B(i)}}{u_i} = \frac{x_{B(\ell)}}{u_\ell} = 0$ if some basic variable $x_{B(\ell)}$ is zero and $d_{B(\ell)} < 0$.
- Thus, $y = x + \theta^* d = x$ and the current basic feasible solution does not change.
- But replacing $A_{B(\ell)}$ with A_j still yields a new basis with associated basic feasible solution $y = x$.

Remark

Even if θ^* is positive, more than one of the original basic variables may become zero at the new point $x + \theta^* d$. Since only one of them leaves the basis, the new basic feasible solution y is degenerate.

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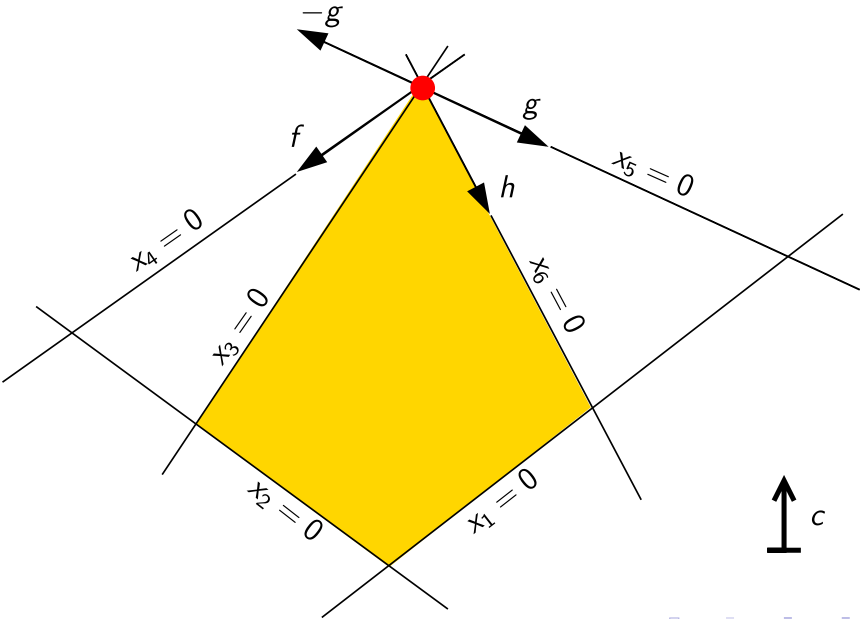
Example



Pivot Selection

Question

How to choose j with $\bar{c}_j < 0$ and ℓ with $\dfrac{x_{B(\ell)}}{u_\ell} = \min\limits_{i:u_i>0} \dfrac{x_{B(i)}}{u_i}$ if several possible choices exists in an iteration of the simplex algorithm?

The choice of j is critical for the overall behavior of the simplex method. Three popular choices are:

- **smallest subscript rule:** choose smallest j with $\bar{c}_j < 0$.
  (very simple; no need to compute entire vector $\bar{c}$; usually leads to many iterations)
- **steepest descent rule:** choose j such that $\bar{c}_j < 0$ is minimal.
  (relatively simple; commonly used for mid-size problems; does not necessarily yield the best neighboring solution)
- **best improvement rule:** choose j such that $\theta^* \bar{c}_j$ is minimal.
  (computationally expensive; used for large problems; usually leads to very few iterations)

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3.3 Implementations of the simplex method

- The revised simplex method
- The full tableau implementation
- Comparison
- Practical performance ehancements

The revised simplex method

Observation

In order to execute one iteration of the simplex method efficiently, it suffices to know $B(1), \ldots, B(m)$, the inverse B^{-1} of the basis matrix and the input data A, b, and c. It is then easy to compute:

$$x_B = B^{-1} b \qquad \overline{c}^T = c^T - c_B{}^T B^{-1} A$$

$$u = B^{-1} A_j \qquad \theta^* = \min_{i: u_i > 0} \frac{x_{B(i)}}{u_i} = \frac{x_{B(\ell)}}{u_\ell}$$

The new basis matrix is then

$$\overline{B} = \left(A_{B(1)} \ldots A_{B(\ell-1)} A_j A_{B(\ell+1)} \ldots A_{B(m)} \right)$$

Question

How to obtain $\overline{B}^{-1}$ efficiently?

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- Notice that $B^{-1}\overline{B} = (e_1 \dots e_{\ell-1}\, u\, e_{\ell+1} \dots e_m)$.
- By elementary linear algebra, $\overline{B}^{-1}$ can be obtained from B^{-1} as follows:
  Multiply the ℓth row of B^{-1} with $1/u_\ell$; then subtract u_i times the resulting ℓth row from the ith row, for $i \neq \ell$.
- These are exactly the elementary row operations needed to turn $B^{-1}\overline{B}$ into the identity matrix!
- Elementary row operations are the same as multiplying the matrix with corresponding elementary matrices from the left hand side.
- Equivalently:

Obtaining $\overline{B}^{-1}$ from B^{-1}

Apply elementary row operations to the matrix $(B^{-1}|u)$ to make the last column equal to the unit vector e_ℓ. The first m columns of the resulting matrix form the inverse $\overline{B}^{-1}$ of the new basis matrix $\overline{B}$.

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