In this section we discuss two pivoting rules that are guaranteed to avoid cycling. These are

- the lexicographic rule
- and Bland's rule.

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Lexicographic order

Definition

- A vector u ∈ ℝⁿ is *lexicographically positive (negative)* if u ≠ 0 and the first nonzero entry of u is positive (negative). Symbolically, we write u > 0 (resp. u < 0).
- A vector $u \in \mathbb{R}^n$ is *lexicographically larger (smaller)* than a vector $v \in \mathbb{R}^n$ if $u \neq v$ and $u v \stackrel{L}{>} 0$ (resp. $u v \stackrel{L}{<} 0$). We write $u \stackrel{L}{>} v$ (resp. $u \stackrel{L}{<} v$).

Example

$$(0, 2, 3, 0)^T \stackrel{L}{>} (0, 2, 1, 4)^T$$

 $(0, 4, 5, 0)^T \stackrel{L}{<} (1, 2, 1, 2)^T$

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We describe the lexicographic pivoting rule in the full tableau implementation.

Lexicographic pivoting rule

- Choose an arbitrary column A_j with $\overline{c}_j < 0$ to enter the basis. Let $u := B^{-1}A_j$ be the *j*th column of the tableau.
- ② For each *i* with $u_i > 0$, divide the *i*th row of the tableau by u_i and choose the lexicographically smallest row ℓ . Then the ℓ th basic variable $x_{B(\ell)}$ exits the basis.

Remark

The lexicographic pivoting rule always leads to a unique choice for the exiting variable. Otherwise two rows of $B^{-1}A$ would have to be linearly dependent which contradicts our assumption on the matrix A.

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The lexicographic pivoting rule (cont.)

Theorem

Suppose that the simplex algorithm starts with lexicographically positive rows 1, . . . , m in the simplex tableau. Suppose that the lexicographic pivoting rule is followed. Then:

- Rows 1, ..., m of the simplex tableau remain lexicographically positive throughout the algorithm.
- ② The zeroth row strictly increases lexicographically at each iteration.
- **③** The simplex algorithm terminates after a finite number of iterations.

Proof.

See eChalk...

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Remarks on the lexicographic pivoting rule

- The lexicographic pivoting rule was derived by considering s small perturbation of the right hand side vector *b* leading to a nondegenerate problem (see exercises).
- The lexicographic pivoting rule can also be used in conjunction with the revised simplex method, provided that B⁻¹ is computed explicitly (this is not the case in sophisticated implementations).
- The assumption in the theorem on the lexicographically positive rows in the tableau can be made without loss of generality: Rearrange the columns of A such that the basic columns (forming the identity matrix in the tableau) come first. Since the zeroth column is nonnegative for a basic feasible solution, all rows are lexicographically positive.

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Bland's rule

Smallest subscript pivoting rule (Bland's rule)

- Choose the column A_j with $\overline{c}_j < 0$ and j minimal to enter the basis.
- Among all basic variables x_i that could exit the basis, select the one with smallest i.

Theorem (without proof)

The simplex algorithm with Bland's rule terminates after a finite number of iterations.

Remark

Bland's rule is compatible with an implementation of the revised simplex method in which the reduced costs of the nonbasic variables are computed one at a time, in the natural order, until a negative one is discovered.

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So far we always assumed that the simplex algorithm starts with a basic feasible solution. In this section we discuss how such a solution can be obtained.

- Introducing artificial variables
- The two-phase simplex method
- The big-*M* method

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Introducing artificial variables

Example LP

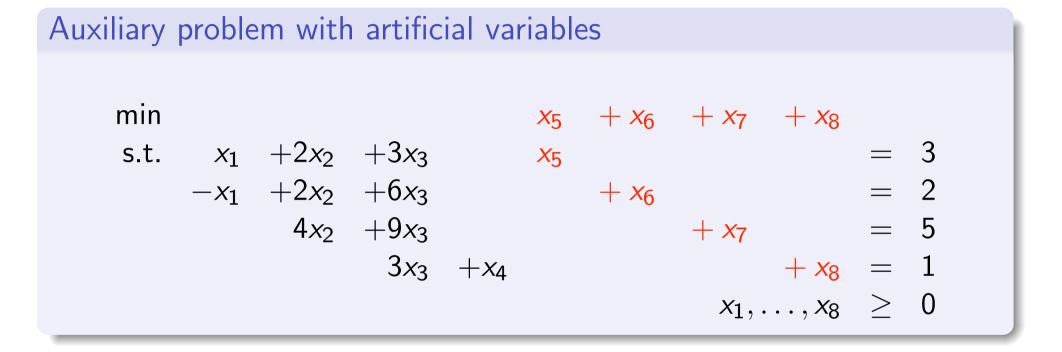
Auxiliary problem with artificial variables

Introducing artificial variables

Example LP

Auxiliary problem with artificial variables

The auxiliary problem



Observation

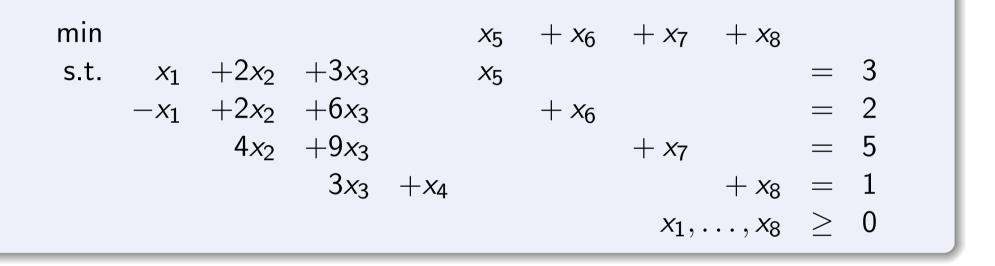
x = (0, 0, 0, 0, 3, 2, 5, 1) is a basic feasible solution for this problem with basic variables (x_5, x_6, x_7, x_8) . We can form the initial tableau.

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The auxiliary problem

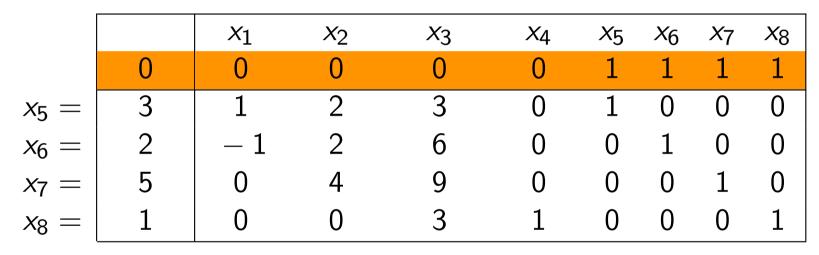
Auxiliary problem with artificial variables



Observation

x = (0, 0, 0, 0, 3, 2, 5, 1) is a basic feasible solution for this problem with basic variables (x_5, x_6, x_7, x_8) . We can form the initial tableau.

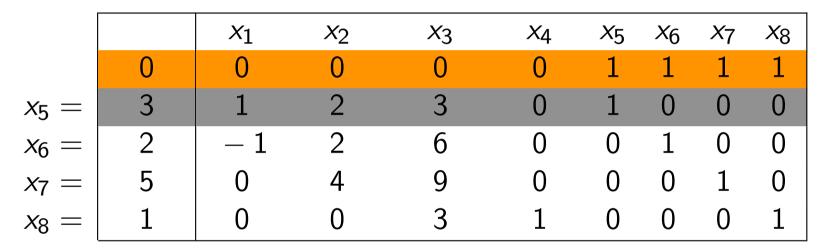
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Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

Now we can proceed as seen before...

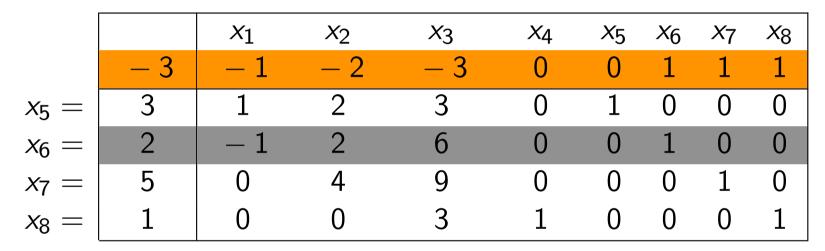
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Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

Now we can proceed as seen before...

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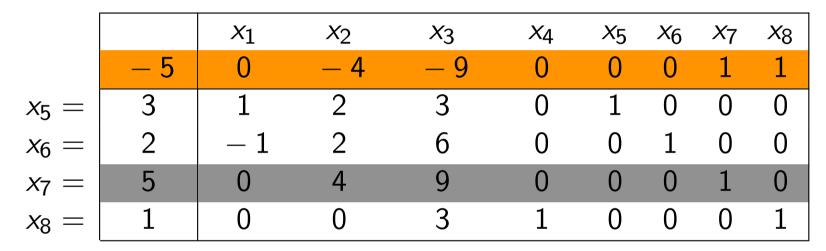
Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

Now we can proceed as seen before...

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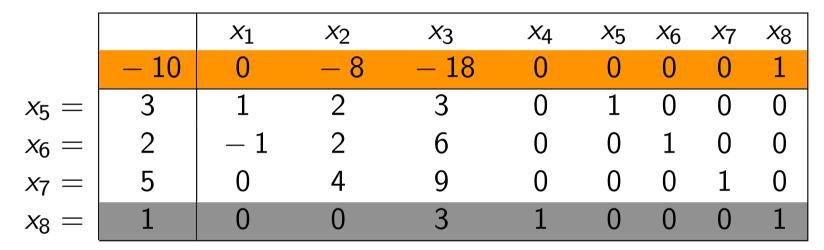


Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

Now we can proceed as seen before...

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Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

Now we can proceed as seen before...

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		<i>x</i> ₁	<i>x</i> ₂	<i>X</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> ₆	<i>X</i> 7	<i>x</i> ₈
	-11	0	-8	-21	-1	0	0	0	0
$x_{5} =$	3	1	2	3	0	1	0	0	0
$x_{6} =$	2	-1	2	6	0	0	1	0	0
<i>x</i> ₇ =	5	0	4	9	0	0	0	1	0
<i>x</i> ₈ =	1	0	0	3	1	0	0	0	1

Calculate reduced costs by eliminating the nonzero-entries for the basis-variables.

Now we can proceed as seen before...

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		<i>x</i> ₁	<i>x</i> ₂	<i>x</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> ₆	<i>X</i> 7	<i>x</i> ₈
	-11	0	-8	-21	-1	0	0	0	0
$x_{5} =$	3	1	2	3	0	1	0	0	0
$x_{6} =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
<i>x</i> ₈ =	1	0	0	3	1	0	0	0	1

Basis change: x_4 enters the basis, x_8 exits.

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		<i>x</i> ₁	<i>x</i> ₂	<i>X</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> ₆	<i>X</i> 7	<i>x</i> 8
	-11	0	-8	-21	-1	0	0	0	0
$x_{5} =$	3	1	2	3	0	1	0	0	0
$x_{6} =$	2	-1	2	6	0	0	1	0	0
<i>x</i> ₇ =	5	0	4	9	0	0	0	1	0
<i>x</i> ₈ =	1	0	0	3	1	0	0	0	1

Basis change: x_4 enters the basis, x_8 exits.

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		<i>x</i> ₁	<i>x</i> ₂	<i>x</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> 6	<i>X</i> 7	<i>X</i> 8
	-10	0	-8	-18	0	0	0	0	1
$x_5 =$	3	1	2	3	0	1	0	0	0
$x_{6} =$	2	-1	2	6	0	0	1	0	0
$x_7 =$	5	0	4	9	0	0	0	1	0
<i>x</i> ₄ =	1	0	0	3	1	0	0	0	1

Basis change: x_3 enters the basis, x_4 exits.

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		<i>x</i> ₁	<i>x</i> ₂	<i>X</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> 6	<i>X</i> 7	<i>x</i> 8
	-10	0	-8	-18					
$x_{5} =$	3	1	2	3	0	1	0	0	0
$x_{6} =$	2	-1	2	6	0	0	1	0	0
$x_{7} =$	5	0	4	9	0	0	0	1	0
$x_{4} =$	1	0	0	3	1	0	0	0	1

Basis change: x_3 enters the basis, x_4 exits.

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		<i>x</i> ₁	<i>x</i> ₂	<i>x</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> 6	<i>X</i> 7	<i>x</i> 8
	_4	0	-8	0	6	0	0	0	7
$x_{5} =$	2	1	2	0	-1	1	0	0	-1
$x_{6} =$	0	-1	2	0	-2	0	1	0	-2
$x_7 =$	2	0	4	0	-3	0	0	1	-3
<i>x</i> ₃ =	1/3	0	0	1	1/3	0	0	0	1/3

Basis change: x_2 enters the basis, x_6 exits.

		<i>x</i> ₁	<i>x</i> ₂	<i>x</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> ₆	<i>X</i> 7	<i>x</i> ₈
	_4	0	-8	0	6	0	0	0	7
$x_{5} =$	2	1	2	0	-1	1	0	0	-1
$x_{6} =$	0	-1	2 4	0	-2		1		
$x_7 =$	2	0					0		
<i>x</i> ₃ =	1/3	0	0	1	1/3	0	0	0	1/3

Basis change: x_2 enters the basis, x_6 exits.

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		<i>x</i> ₁	<i>x</i> ₂	<i>x</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> ₆	<i>X</i> 7	<i>x</i> 8
	_4	-4	0	0	-2	0	4	0	-1
$x_5 =$	2	2	0	0	1	1	-1	0	1
$x_2 =$	0	-1/2	1	0	-1	0	1/2	0	-1
				1					
<i>x</i> ₃ =	1/3	0	0	1	1/3	0	0	0	1/3

Basis change: x_1 enters the basis, x_5 exits.

		<i>x</i> ₁	<i>x</i> ₂	<i>x</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> ₆	<i>X</i> 7	<i>x</i> ₈
	_4	-4	0	0	-2	0	4	0	-1
$x_{5} =$	2	2	0	0	1	1	-1	0	1
$x_2 =$	0	-1/2	1	0	-1	0	1/2	0	-1
$x_7 =$	2	0	0	1	0	-2	1	1	$1 \mid$
<i>x</i> ₃ =	1/3	0	0	1	1/3	0	0	0	1/3

Basis change: x_1 enters the basis, x_5 exits.

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		<i>x</i> ₁	<i>x</i> ₂	<i>X</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> 6	<i>X</i> 7	<i>x</i> ₈
	0	0	0	0	0	2	2	0	1
$x_1 =$	1	1	0	0	1/2	1/2	-1/2	0	1/2
$x_2 =$	1/2	0	1	0	-3/4	1/4	1/4	0	-3/4
$x_7 =$	0	0	0	0	0	-1	-1	1	0
<i>x</i> ₃ =	1/3	0	0	1	1/3	0	0	0	1/3

Basic feasible solution for auxiliary problem with (auxiliary) cost value $0 \Rightarrow$ Also feasible for the original problem - but not (yet) basic.

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		<i>x</i> ₁	<i>x</i> ₂	<i>x</i> 3	<i>X</i> 4	<i>X</i> 5	x ₆	<i>X</i> 7	<i>x</i> ₈
	0	0	0	0	0	2	2	0	1
$x_1 =$	1	1	0	0	1/2	1/2	-1/2	0	1/2
$x_2 =$	1/2	0	1	0	-3/4	1/4	1/4	0	-3/4
$x_7 =$	0	0	0	0	0	-1	-1	1	0
<i>x</i> ₃ =	1/3	0	0	1	1/3	0	0	0	1/3

Basic feasible solution for auxiliary problem with (auxiliary) cost value 0 \Rightarrow Also feasible for the original problem - but not (yet) basic.

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		<i>x</i> ₁	<i>x</i> ₂	<i>x</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> 6	<i>X</i> 7	<i>x</i> 8
	0	0				2	2	0	1
$x_1 =$	1	1	0	0	1/2	1/2	-1/2	0	1/2
									-3/4
$x_7 =$	0	0	0	0	0	-1	-1	1	0
<i>x</i> ₃ =	1/3	0	0	1	1/3	0	0	0	1/3

Basic feasible solution for auxiliary problem with (auxiliary) cost value $0 \Rightarrow$ Also feasible for the original problem - but not (yet) basic.

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Getting a basis for the original problem

		<i>x</i> ₁	<i>x</i> ₂	<i>x</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> 6	<i>X</i> 7	<i>x</i> 8
	0	0	0	0	0	2	2	0	1
$x_1 =$	1	1	0	0	1/2	1/2	-1/2	0	1/2
$x_2 =$	1/2	0	1	0	-3/4	1/4	1/4	0	-3/4
	0				0				
$x_3 = $	1/3	0	0	1	1/3	0	0	0	1/3

Observation

Restricting the tableau to the original variables, we get a zero-row.

Thus the original equations are linearily dependent.

 \rightarrow We can remove the third row.

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		<i>x</i> ₁	<i>x</i> ₂	<i>x</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> 6	<i>X</i> 7	<i>x</i> 8
	0	0	0	0	0	2	2	0	1
$x_1 =$	1				1/2	1/2	-1/2	0	1/2
$x_2 =$	1/2	0	1	0	-3/4	1/4	1/4	0	-3/4
$x_7 =$	0	0	0	0	0	-1	-1	1	0
<i>x</i> ₃ =	1/3	0	0	1	1/3	0	0	0	1/3

Observation

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Thus the original equations are linearily dependent.

 \rightarrow We can remove the third row.

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		<i>x</i> ₁	<i>x</i> ₂	<i>x</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> 6	<i>X</i> 7	<i>x</i> ₈
	0	0	0	0	0	2	2	0	1
$x_1 =$	1	1	0	0	1/2	1/2	-1/2	0	1/2
$x_2 =$	1/2	0	1	0	-3/4	1/4	1/4	0	-3/4
$x_7 =$	0	0	0	0	0	-1	-1	1	0
<i>x</i> ₃ =	1/3	0	0	1	1/3	0	0	0	1/3

Observation

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		<i>x</i> ₁	<i>x</i> ₂	<i>x</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> ₆	<i>X</i> 7	<i>x</i> ₈
	0	0	0	0	0	2	2	0	1
$x_1 =$	1	1	0	0	1/2	1/2	-1/2	0	1/2
$x_2 =$	1/2	0	1	0	-3/4	1/4	1/4	0	-3/4
$x_7 =$	0	0	0	0	0	-1	-1	1	0
<i>x</i> ₃ =	1/3	0	0	1	1/3	0	0	0	1/3

Observation

Restricting the tableau to the original variables, we get a zero-row.

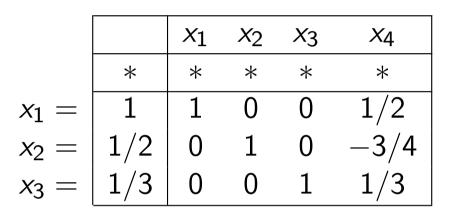
Thus the original equations are linearily dependent.

 \rightarrow We can remove the third row.

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We finally obtain a basic feasible solution for the original problem. After computing the reduced costs for this basis (as seen in the beginning), the simplex method can start with its typical iterations.

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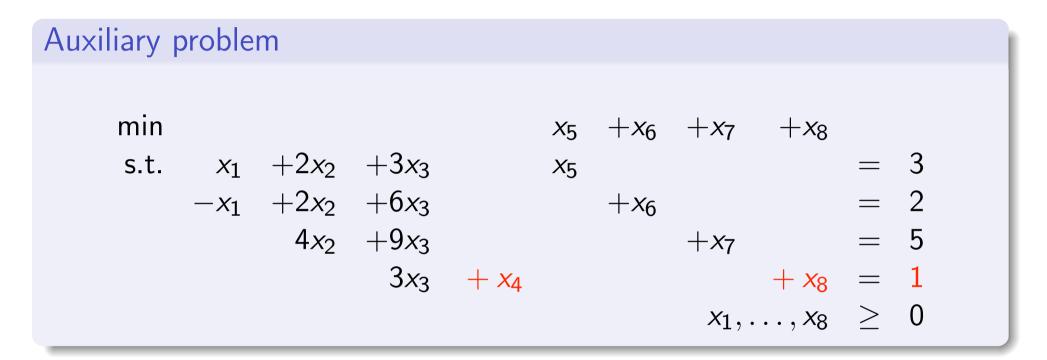
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We finally obtain a basic feasible solution for the original problem. After computing the reduced costs for this basis (as seen in the beginning), the simplex method can start with its typical iterations.

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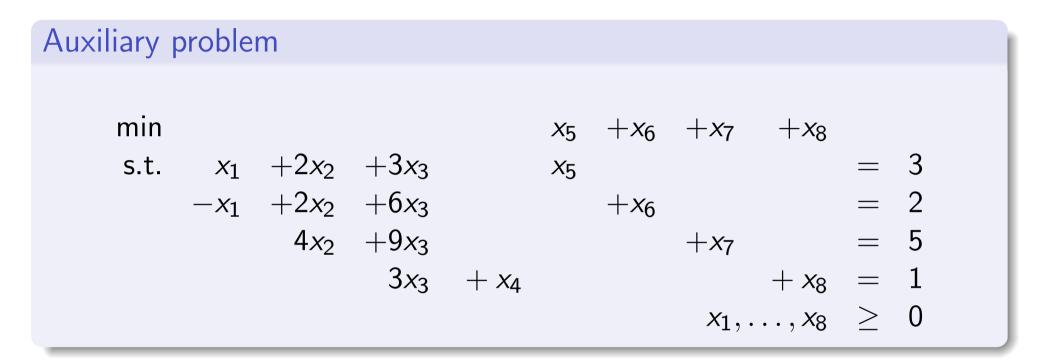
Omitting artificial variables



Artificial variable x_8 could have been omitted by setting x_4 to 1 in the initial basis. This is possible as x_4 does only appear in one constraint. Generally, this can be done e.g. with all slack variables that have nonnegative right hand sides.

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Omitting artificial variables



Artificial variable x_8 could have been omitted by setting x_4 to 1 in the initial basis. This is possible as x_4 does only appear in one constraint. Generally, this can be done e.g. with all slack variables that have nonnegative right hand sides.

Given: LP in standard form: min $c^T x$ s.t. Ax = b, $x \ge 0$

- ① Transform problem such that $b \ge 0$ (multiply constraints by -1).
- 2 Introduce artificial variables y_1, \ldots, y_m and solve auxiliary problem

$$\min \sum_{i=1}^m y_i \quad \text{s.t. } Ax + I_m y = b, \ x, y \ge 0 \ .$$

- If optimal cost is positive, then STOP (original LP is infeasible).
- If no artificial variable is in final basis, eliminate artificial variables and columns and STOP (feasible basis for original LP has been found).
- If ℓth basic variable is artificial, find $j \in \{1, ..., n\}$ with ℓth entry in $B^{-1}A_j$ nonzero. Use this entry as pivot element and replace ℓth basic variable with x_j .

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• If no such $j \in \{1, ..., n\}$ exists, eliminate ℓ th row (constraint).

Given: LP in standard form: $\min c^T x$ s.t. $Ax = b, x \ge 0$

- Transform problem such that $b \ge 0$ (multiply constraints by -1).
- 2 Introduce artificial variables y_1, \ldots, y_m and solve auxiliary problem

$$\min \sum_{i=1}^m y_i \quad \text{s.t. } Ax + I_m y = b, \ x, y \ge 0 \ .$$

If optimal cost is positive, then STOP (original LP is infeasible).

- If no artificial variable is in final basis, eliminate artificial variables and columns and STOP (feasible basis for original LP has been found).
- If ℓth basic variable is artificial, find $j \in \{1, ..., n\}$ with ℓth entry in $B^{-1}A_j$ nonzero. Use this entry as pivot element and replace ℓth basic variable with x_j .
- If no such $j \in \{1, ..., n\}$ exists, eliminate ℓ th row (constraint).

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Given: LP in standard form: min $c^T x$ s.t. Ax = b, $x \ge 0$

- **①** Transform problem such that $b \ge 0$ (multiply constraints by -1).
- ② Introduce artificial variables y_1, \ldots, y_m and solve auxiliary problem

$$\min \sum_{i=1}^m y_i \quad \text{s.t. } Ax + I_m y = b, \ x, y \ge 0$$

- If optimal cost is positive, then STOP (original LP is infeasible).
- If no artificial variable is in final basis, eliminate artificial variables and columns and STOP (feasible basis for original LP has been found).
- **○** If ℓ th basic variable is artificial, find $j \in \{1, ..., n\}$ with ℓ th entry in $B^{-1}A_j$ nonzero. Use this entry as pivot element and replace ℓ th basic variable with x_j .

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If no such $j \in \{1, ..., n\}$ exists, eliminate ℓ th row (constraint).

Given: LP in standard form: min $c^T x$ s.t. Ax = b, $x \ge 0$

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- ⑤ If ℓ th basic variable is artificial, find $j \in \{1, ..., n\}$ with ℓ th entry in $B^{-1}A_j$ nonzero. Use this entry as pivot element and replace ℓ th basic variable with x_j .

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• If no such $j \in \{1, \ldots, n\}$ exists, eliminate ℓ th row (constraint).

Given: LP in standard form: min $c^T x$ s.t. Ax = b, $x \ge 0$

- **①** Transform problem such that $b \ge 0$ (multiply constraints by -1).
- ② Introduce artificial variables y_1, \ldots, y_m and solve auxiliary problem

$$\min \sum_{i=1}^m y_i \quad \text{s.t. } Ax + I_m y = b, \ x, y \ge 0$$

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Alternative idea: Combine the two phases into one by introducing sufficiently large penalty costs for artificial variables.



min
$$\sum_{i=1}^{n} c_i x_i + M \sum_{j=1}^{m} y_j$$

s.t. $Ax + y = b$.
 $x, y \ge 0$

If M is sufficiently large and the original program has a feasible solution, all artificial variables will be driven to zero by the simplex method.

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This way, the LP

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Observation

Initially, *M* only occurs in the zeroth row. As the zeroth row never becomes pivot row, this property is maintained while the simplex method is running.

All we need to have is an order on all values that can appear as reduced cost coefficients.

Order on cost coefficients

 $aM + b < cM + d : \Leftrightarrow (a < c) \lor (a = c \land b < c)$

In particular, -aM + b < 0 < aM + b for any positive *a* and arbitrary *b*, and we can decide whether a cost coefficient is negative or not.

 \rightarrow There is no need to give *M* a fixed numerical value.

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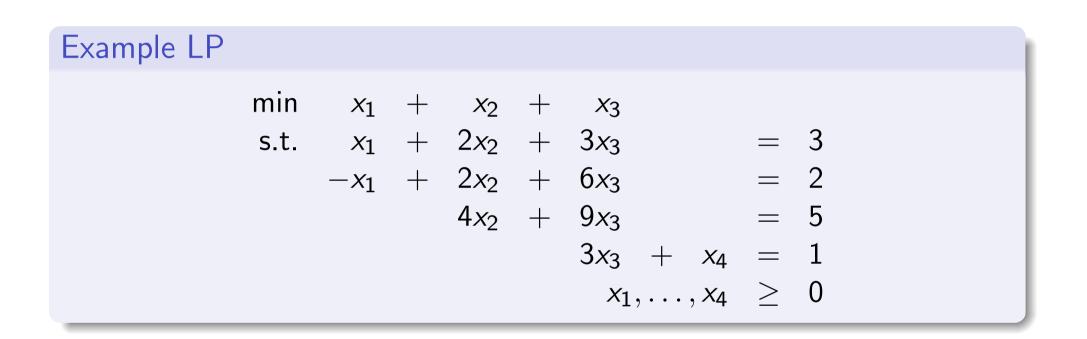
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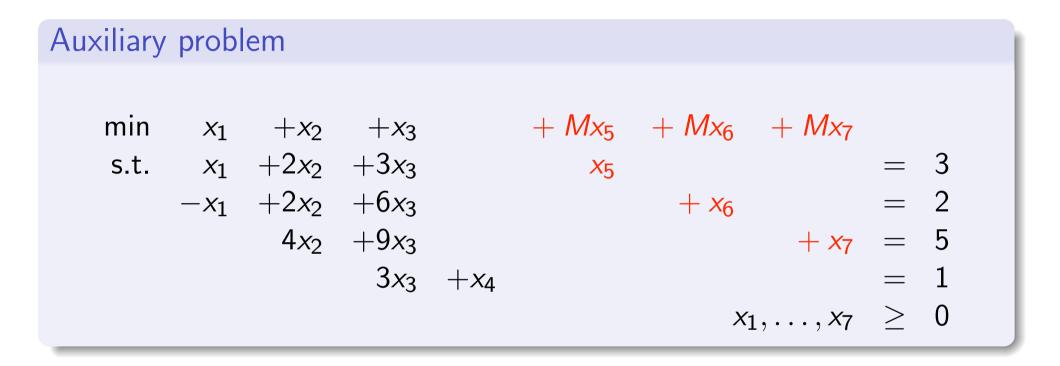
Example



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Introducing artificial variables and M



Note that this time the unnecessary artificial variable x_8 has been omitted. We start off with $(x_5, x_6, x_7, x_4) = (3, 2, 5, 1)$.

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	<i>x</i> ₁	<i>x</i> ₂	<i>x</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> ₆	<i>X</i> 7
0	1	1	1	0	M	M	M
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Compute reduced costs by eliminating the nonzero entries for the basic variables.

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	<i>x</i> ₁	<i>x</i> ₂	<i>x</i> ₃	<i>X</i> 4	<i>X</i> 5	<i>x</i> ₆	<i>X</i> 7
0	1	1	1	0	M	M	M
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
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	<i>x</i> ₁	<i>x</i> ₂	<i>x</i> ₃	<i>X</i> 4	<i>X</i> 5	<i>x</i> ₆	<i>X</i> 7
— 3 <i>M</i>	-M + 1	-2M + 1	-3M + 1	0	0	M	M
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

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	<i>x</i> ₁	<i>x</i> ₂	<i>x</i> 3	<i>x</i> 4	<i>X</i> 5	<i>x</i> ₆	<i>X</i> 7
-5M	1	-4M + 1	-9M + 1	0	0	0	M
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

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	<i>x</i> ₁	<i>x</i> ₂	<i>X</i> 3	<i>x</i> 4	<i>X</i> 5	<i>x</i> ₆	<i>X</i> 7
-10M	1	-8M + 1	-18M + 1	0	0	0	0
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
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-10M	1	-8M + 1	-18M + 1	0	0	0	0
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First iteration

	<i>x</i> ₁	<i>x</i> ₂	<i>x</i> ₃	<i>X</i> 4	<i>X</i> 5	<i>x</i> 6	<i>X</i> 7
-10M	1	-8M + 1	-18M + 1	0	0	0	0
3	1	2	3	0	1	0	0
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5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Reduced costs for x_2 and x_3 are negative. Basis change: x_3 enters the basis, x_4 leaves.

Martin Skutella (TU Berlin)Linear and Integer Programming (ADM II)WS 2007/08

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First iteration

	<i>x</i> ₁	<i>x</i> ₂	<i>X</i> 3	<i>x</i> 4	<i>X</i> 5	<i>x</i> ₆	<i>X</i> 7
-10M	1	-8M + 1	-18M + 1	0	0	0	0
3	1	2	3	0	1	0	0
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Reduced costs for x_2 and x_3 are negative.

Basis change: x_3 enters the basis, x_4 leaves.

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First iteration

	<i>x</i> ₁	<i>x</i> ₂	<i>x</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> ₆	<i>X</i> 7
-10M	1	-8M + 1	-18M + 1	0	0	0	0
3	1	2	3	0	1	0	0
2	-1	2	6	0	0	1	0
5	0	4	9	0	0	0	1
1	0	0	3	1	0	0	0

Reduced costs for x_2 and x_3 are negative. Basis change: x_3 enters the basis, x_4 leaves.

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Second iteration

	<i>x</i> ₁	<i>x</i> ₂	<i>X</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> 6	<i>X</i> 7
-4M - 1/3	1	-8M + 1	0	6M - 1/3	0	0	0
2	1	2	0	-1	1	0	0
0	-1	2	0	-2	0	1	0
2	0	4	0	-3	0	0	1
1/3	0	0	1	1/3	0	0	0

Basis change: x_2 enters the basis, x_6 leaves.

Second iteration

	<i>x</i> ₁	<i>x</i> ₂	<i>X</i> 3	<i>X</i> 4	<i>X</i> 5	<i>x</i> 6	<i>X</i> 7
-4M - 1/3	1	-8M + 1	0	6M - 1/3	0	0	0
2	1	2	0	-1	1	0	0
0	-1	2	0	-2	0	1	0
2	0	4	0	-3	0	0	1
1/3	0	0	1	1/3	0	0	0

Basis change: x_2 enters the basis, x_6 leaves.

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Third iteration

	x ₁	<i>x</i> ₂	<i>X</i> 3	<i>x</i> 4	<i>X</i> 5	<i>x</i> ₆	<i>X</i> 7
-4M - 1/3	-4M + 3/2	0	0	-2M + 2/3	0	4M - 1/2	0
2	2	0	0	1	1	-1	0
0	-1/2	1	0	-1	0	1/2	0
2	2	0	0	1	0	-2	1
1/3	0	0	1	1/3	0	0	0

Basis change: x_1 enters the basis, x_5 leaves.

Third iteration

	x ₁	<i>x</i> ₂	<i>X</i> 3	<i>x</i> 4	<i>X</i> 5	<i>x</i> ₆	<i>X</i> 7
-4M - 1/3	-4M + 3/2	0	0	-2M + 2/3	0	4M - 1/2	0
2	2	0	0	1	1	-1	0
0	-1/2	1	0	-1	0	1/2	0
2	2	0	0	1	0	-2	1
1/3	0	0	1	1/3	0	0	0

Basis change: x_1 enters the basis, x_5 leaves.

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Fourth iteration

	<i>x</i> ₁	<i>x</i> ₂	<i>X</i> 3	X ₄	<i>X</i> 5	x ₆	<i>X</i> 7
-11/6	0	0	0	-1/12	2M - 3/4	2M + 1/4	0
21	1	0	0	1/2	1/2	-1/2	0
1/2	0	1	0	-3/4	1/4	1/4	0
0	0	0	0	0	-1	-1	1
1/3	0	0	1	1/3	0	0	0

Note that all artificial variables have already been driven to 0.

Basis change: x_4 enters the basis, x_3 leaves.

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Fourth iteration

	<i>x</i> ₁	<i>x</i> ₂	X3	<i>X</i> 4	<i>X</i> 5	x ₆	<i>X</i> 7
-11/6	0	0	0	-1/12	2M - 3/4	2M + 1/4	0
21	1	0	0	1/2	1/2	-1/2	0
1/2	0	1	0	-3/4	1/4	1/4	0
0	0	0	0	0	-1	-1	1
1/3	0	0	1	1/3	0	0	0

Note that all artificial variables have already been driven to 0.

Basis change: x_4 enters the basis, x_3 leaves.

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Fourth iteration

	<i>x</i> ₁	<i>x</i> ₂	<i>X</i> 3	X ₄	<i>X</i> 5	x ₆	<i>X</i> 7
-11/6	0	0	0	-1/12	2M - 3/4	2M + 1/4	0
21	1	0	0	1/2	1/2	-1/2	0
1/2	0	1	0	-3/4	1/4	1/4	0
0	0	0	0	0	-1	-1	1
1/3	0	0	1	1/3	0	0	0

Note that all artificial variables have already been driven to 0.

Basis change: x_4 enters the basis, x_3 leaves.

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Fifth iteration

	<i>x</i> ₁	<i>x</i> ₂	<i>X</i> 3	Х4	<i>X</i> 5	x ₆	<i>X</i> 7
-7/4	0	0	1/4	0	2M - 3/4	2M + 1/4	0
1/2	1	0	-3/2	0	1/2	-1/2	0
5/4	0	1	9/4	0	1/4	1/4	0
0	0	0	0	0	-1	-1	1
1	0	0	3	1	0	0	0

We now have an optimal solution of the auxiliary problem, as all costs are nonnegative (M presumed large enough).

By elimiating the third row as in the previous example, we get a basic feasible and also optimal solution to the original problem.

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Fifth iteration

	<i>x</i> ₁	<i>x</i> ₂	<i>X</i> 3	<i>X</i> 4	<i>X</i> 5	x ₆	<i>X</i> 7
-7/4	0	0	1/4	0	2M - 3/4	2M + 1/4	0
1/2	1	0	-3/2	0	1/2	-1/2	0
5/4	0	1	9/4	0	1/4	1/4	0
0	0	0	0	0	-1	-1	1
1	0	0	3	1	0	0	0

We now have an optimal solution of the auxiliary problem, as all costs are nonnegative (M presumed large enough).

By elimiating the third row as in the previous example, we get a basic feasible and also optimal solution to the original problem.

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Fifth iteration

	<i>x</i> ₁	<i>x</i> ₂	<i>X</i> 3	<i>X</i> 4	<i>X</i> 5	x ₆	<i>X</i> 7
-7/4	0	0	1/4	0	2M - 3/4		0
1/2	1	0	-3/2	0	1/2	-1/2	0
5/4	0	1	9/4	0	1/4	1/4	0
0	0	0	0	0	-1	-1	1
1	0	0	3	1	0	0	0

We now have an optimal solution of the auxiliary problem, as all costs are nonnegative (M presumed large enough).

By elimiating the third row as in the previous example, we get a basic feasible and also optimal solution to the original problem.

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