## 3.7 Computational efficiency of the simplex method

### Observation

The computational efficiency of the simplex method is determined by

1. the computational effort of each iteration;
2. the number of iterations.

### Question

How many iterations are needed in the worst case?

### Idea for negative answer (lower bound)

Describe

- a polyhedron with an exponential number of vertices;
- a path that visits all vertices and always moves from a vertex to an adjacent one that has lower costs.



## Computational efficiency of the simplex method

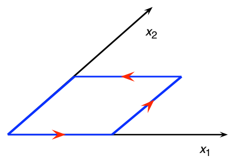
### Unit cube

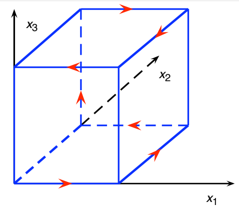
Consider the unit cube in $\mathbb{R}^n$, defined by the constraints

$$0 \leq x_i \leq 1, \quad i = 1, \ldots, n$$

The unit cube has

- $2^n$ vertices;
- a *spanning path*, i.e. a path traveling the edges of the cube visiting each vertex exactly once.





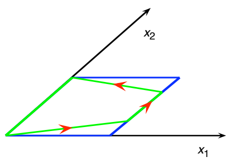

## Computational efficiency of the simplex method

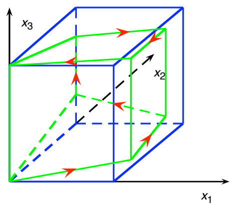
### Klee-Minty cube

Consider a perturbation of the unit cube in $\mathbb{R}^n$, defined by the constraints

$$0 \leq x_1 \leq 1,$$
$$\epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \ldots, n$$

for some $\epsilon \in (0, 1/2)$.






## Computational efficiency of the simplex method

### Klee-Minty cube

$$0 \leq x_1 \leq 1,$$
$$\epsilon x_{i-1} \leq x_i \leq 1 - \epsilon x_{i-1}, \quad i = 2, \ldots, n, \epsilon \in (0, 1/2)$$

### Theorem

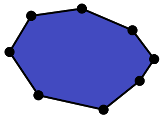
*Consider the linear programming problem of minimizing $-x_n$ subject to the constraints above. Then:*

1. *The feasible set has $2^n$ vertices.*
2. *The vertices can be ordered so that each one is adjacent to and has lower cost than the previous one.*
3. *There exists a pivoting rule under which the simplex method requires $2^n - 1$ changes of basis before it terminates.*

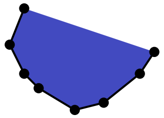
## The diameter of polyhedra

### Definition

- The *distance* $d(x, y)$ between two vertices $x, y$ is the minimum number of edges required to reach $y$ starting from $x$.
- The *diameter* $D(P)$ of polyhedron $P$ is the maximum $d(x, y)$ over all pairs of vertices $(x, y)$.
- $\Delta(n, m)$ is the maximum $D(P)$ over *all bounded* polyhedra in $\mathbb{R}^n$ that are represented in terms of $m$ inequality constraints.
- $\Delta_u(n, m)$ is the maximum $D(P)$ over *all* polyhedra in $\mathbb{R}^n$ that are represented in terms of $m$ inequality constraints.



$\Delta(2, 8) = \left\lfloor \frac{8}{2} \right\rfloor = 4$



$\Delta_u(2, 8) = 8 - 2 = 6$



## The Hirsch Conjecture

### Observation

The diameter of the feasible set in a linear programming problem is a lower bound on the number of steps required by the simplex method, no matter which pivoting rule is being used.

### Hirsch Conjecture

$$\Delta(n, m) \leq m - n$$

### Known bounds

- Lower bounds: $\Delta_u(n, m) \geq m - n + \left\lfloor \frac{n}{5} \right\rfloor$
- Upper bounds:

$$\Delta(n, m) \leq \Delta_u(n, m) < m^{1+\log_2 n} = (2n)^{\log_2 m}$$



## Average case behavior of the simplex method

### Remark

- Despite the exponential lower bounds on the worst case behavior of the simplex method (Klee-Minty cubes etc.), the simplex method usually behaves well in practice.
- The number of iterations is “typically” $O(m)$.
- There have been several attempts to explain this phenomenon from more a theoretical point of view.
- These results say that “on average” the number of iterations is $O(\cdot)$ (usually polynomial).
- One main difficulty is to come up with a meaningful and, at the same time, manageable definition of the term “on average”.