Chapter 4: Duality Theory

4.1 Motivation

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax \geq b \\ & x \geq 0 \end{aligned} \qquad A \in \mathbb{R}^{m \times n}$$

Try to derive lower bounds on the value of an optimum solution. Use the information that $Ax \geq b$ for any feasible solution x.

For $p \in \mathbb{R}^m$ with $p \geq 0$: $Ax \geq b \Rightarrow (p^T \cdot A) \cdot x \geq p^T \cdot b$

If $c^T \geq p^T \cdot A$, then $c^T x \geq (p^T A) \cdot x \geq p^T b$ for all feasible solutions to our LP.

How to find the best (largest) lower bound in this way?

$$\begin{aligned} \max\ & p^T b \\ \text{s.t. } & p^T \cdot A \leq c^T \\ & p \geq 0 \end{aligned} \qquad \Leftrightarrow \qquad \begin{aligned} \max\ & b^T \cdot p \\ \text{s.t. } & A^T \cdot p \leq c \\ & p \geq 0 \end{aligned}$$

This LP is the dual linear program of our initial LP.

More general:

4.2 The dual problem

$$\min\ c^T x$$

$$\begin{aligned}
\text{s.t.}\quad & a_i^T \cdot x \geq b_i && \text{for } i \in M_1 \\
& a_i^T \cdot x \leq b_i && \text{for } i \in M_2 \\
& a_i^T x = b_i && \text{for } i \in M_3 \\
& x_j \geq 0 && \text{for } j \in N_1 \\
& x_j \leq 0 && \text{for } j \in N_2 \\
& x_j \text{ free} && \text{for } j \in N_3
\end{aligned}$$

Lower bound on $c^T x$ can be obtained by choosing p_i, $i \in M_1 \cup M_2 \cup M_3$ with:

$$\boxed{\max\ p^T \cdot b}$$

$$\begin{aligned}
\text{s.t.}\quad & p_i \geq 0 && \text{for } i \in M_1 \\
& p_i \leq 0 && \text{for } i \in M_2 \\
& p_i \text{ free} && \text{for } i \in M_3 \\
\text{and:}\quad & p^T \cdot A_j \leq c_j && \text{for } j \in N_1 \\
& p^T \cdot A_j \geq c_j && \text{for } j \in N_2 \\
& p^T \cdot A_j = c_j && \text{for } j \in N_3
\end{aligned}$$

This linear program is the dual LP of the primal linear program we started with.

Example:

$\min\ c^T x$	$\max\ p^T b$
s.t. $Ax \geq b$	$p^T A = c^T$
primal LP	$p \geq 0$

Theorem: The dual of the dual LP is the primal LP.

<u>Proof:</u> (for special case only)

Consider primal LP:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{array} \quad \xrightarrow{\text{dualize}} \quad \begin{array}{ll} \max & p^T b \\ \text{s.t.} & p^T A \leq c^T \\ & p \text{ free} \end{array}$$

$$\xrightarrow{\text{equivalent}} \quad \begin{array}{ll} \min & (-b^T) \cdot p \\ \text{s.t.} & A^T \cdot p \leq c \end{array} \quad \xrightarrow{\text{dualize}} \quad \begin{array}{ll} \max & y^T \cdot c \\ \text{s.t.} & y^T \cdot A^T = -b^T \\ & y \leq 0 \end{array}$$

$$\xrightarrow{\text{equivalent}} \quad \begin{array}{ll} \min & c^T \cdot (-y) \\ \text{s.t.} & A \cdot (-y) = b \\ & (-y) \geq 0 \end{array} \quad \longrightarrow \quad \begin{array}{ll} \min & c^T z \\ \text{s.t.} & A \cdot z = b \\ & z \geq 0 \end{array} \quad \square$$