

# LP duality

$$\min c^T x$$

$$a_i^T x \geq b_i$$

$$a_i^T x \leq b_i$$

$$a_i^T x = b_i$$

$$x_j \geq 0$$

$$x_j \leq 0$$

$$x_j \text{ free}$$

$$\max p^T \cdot b$$

$$p_i \geq 0 \quad i \in M_1$$

$$p_i \leq 0 \quad i \in M_2$$

$$p_i \text{ free} \quad i \in M_3$$

$$p^T \cdot A_j \leq c_j \quad j \in N_1$$

$$p^T \cdot A_j \geq c_j \quad j \in N_2$$

$$p^T \cdot A_j = c_j \quad j \in N_3$$

Theorem: The dual of the dual LP is the primal LP.

Theorem: Let  $\pi_1$  and  $\pi_2$  be two LPs where  $\pi_2$  has been obtained from  $\pi_1$  by (several) transformations of the following type

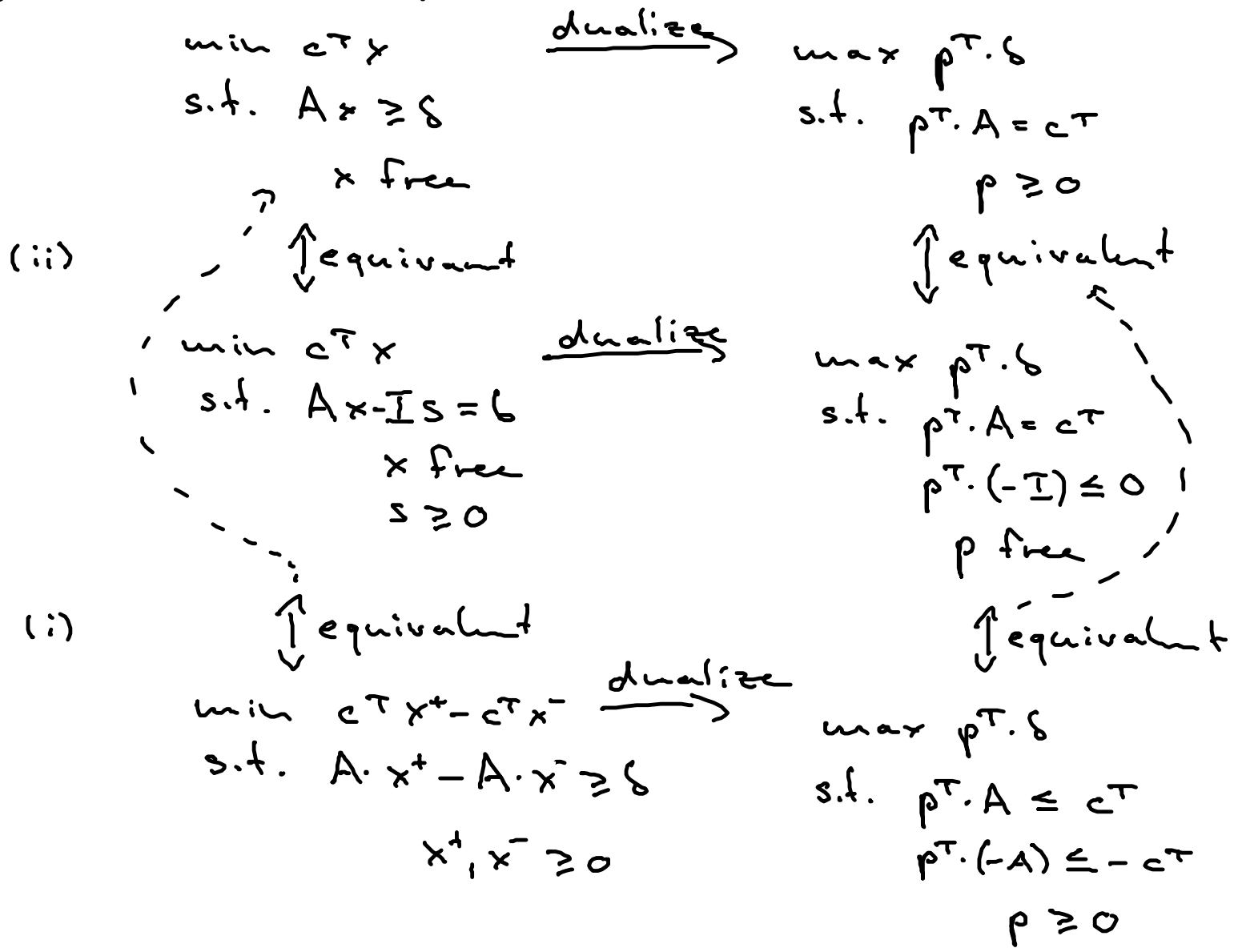
(i) Replace a free variable by difference of two non-negative variables.

(ii) Introducing a slack variable in order to replace an inequality constr. by equal. constr.

(iii) If some row of a feasible equality system is a linear combination of the other rows, eliminate this row.

Then the dual of  $\pi_1$  is equivalent to the dual of  $\pi_2$ .

# Sketch of proof:



(iii) Let  $a_1^T, \dots, a_m^T$  rows of  $A$  and suppose that

$$a_m^T = \sum_{i=1}^{m-1} \gamma_i \cdot a_i^T \quad \text{and} \quad b_m = \sum_{i=1}^{m-1} \gamma_i \cdot b_i$$

Then row  $m$  is redundant and can be eliminated. The dual constraints

$$\sum_{i=1}^m p_i \cdot a_i^T \leq c^T \quad \text{can be rewritten}$$

$$\text{as } \sum_{i=1}^{m-1} (p_i + \gamma_i \cdot p_m) \cdot a_i^T \leq c^T$$

and the dual cost is

$$\sum_{i=1}^m p_i \cdot b_i = \sum_{i=1}^{m-1} (p_i + \gamma_i \cdot p_m) \cdot b_i$$

Let  $q_i := p_i + \gamma_i \cdot p_m$

dual LP  $\overset{\text{equiv.}}{\rightsquigarrow} \max \sum_{i=1}^{m-1} q_i \cdot b_i$   
 s.t.  $\sum_{i=1}^{m-1} q_i \cdot a_i^T \leq c^T \quad \square$

### 4.3 Duality Theorem

Theorem (weak duality):

If  $x$  feasible sol. to primal LP and  $p$  " " " " dual " , then

$$c^T x \geq p^T \cdot b.$$

Proof: Consider a primal-dual pair of LPs.

$$\begin{aligned} \min c^T x \\ \text{s.t. } Ax \geq b \quad (1) \\ x \geq 0 \end{aligned}$$

$$\begin{aligned} \max p^T \cdot b \\ \text{s.t. } p^T A \leq c^T \quad (2) \\ p \geq 0 \end{aligned}$$

$$c^T x \geq \underset{\substack{\uparrow \\ p \text{ fulfills } (2) \\ x \geq 0}}{(p^T \cdot A) \cdot x} = p^T \cdot \underset{\substack{\uparrow \\ x \text{ fulfills } (1) \\ p \geq 0}}{(A \cdot x)} \geq p^T \cdot b \quad \square$$

Corollary: (i) If primal LP is unbounded (i.e. opt. cost =  $-\infty$ ), then the dual LP is infeasible.

(ii) If dual LP is unbounded, then primal LP is infeasible.

Corollary: If  $x, p$  are feasible sol. to the primal and dual LP, respectively, and if  $c^T x = p^T b$ , then  $x$  and  $p$  are optimal solutions.

Theorem (strong duality):

If an LP has an optimal solution, so does its dual and the optimal costs are equal.

Proof: Consider LP in standard form with linearly indep. rows of  $A$ :

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$$\begin{aligned} \max \quad & p^T b \\ \text{s.t.} \quad & p^T A \leq c^T \end{aligned}$$

The simplex method yields opt. basis  $B$  with

$$\bar{c}^T = c^T - c_B^T \cdot B^{-1} \cdot A \geq 0$$

and a corresp. opt. basic feas. sol.  $x$  with

$$x_B = B^{-1} \cdot b.$$

Let  $p^T := c_B^T \cdot B^{-1}$ . Then

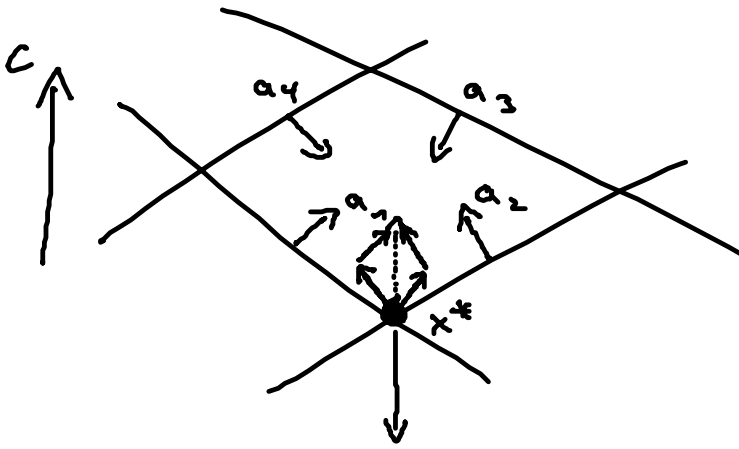
$$p^T \cdot A = c_B^T \cdot B^{-1} \cdot A \leq c^T$$

$$\text{and } p^T \cdot b = c_B^T \cdot (B^{-1} \cdot b) = c_B^T \cdot x_B = c^T \cdot x. \quad \square$$

Interpretation of duality:

$$\begin{aligned} \min c^T x \\ \text{s.t. } Ax \geq b \end{aligned}$$

$$\begin{aligned} \max p^T b \\ \text{s.t. } p^T A = c^T \\ p \geq 0 \end{aligned}$$



$$c^T x^* = p^{*T} \cdot b$$

$$c^T \cdot x^* = p^{*T} \cdot A \cdot x^* \geq p^{*T} \cdot b$$

$$\Rightarrow p_i = 0 \text{ if } a_i^T \cdot x^* > b_i$$

### Complementary slackness:

Theorem: Let  $x, p$  feasible primal and dual solutions, then  $x, p$  are both optimal if and only if:

$$u_i := p_i \cdot (a_i^T x - b_i) = 0 \quad \forall i$$

$$v_j := (c_j - p^T A_j) \cdot x_j = 0 \quad \forall j$$

Proof: Feasibility  $\Rightarrow u_i, v_j \geq 0 \quad \forall i, j$

$$c^T x \geq (p^T A) \cdot x = \underbrace{p^T \cdot (Ax)}_{(*)} \geq \underbrace{p^T b}_{(**)}$$

similar  
for  $v_j$ 's.

$$(*) - (**) = \sum u_i$$

□