# LP duality

$$\min c^T x \qquad\qquad \max p^T \cdot b$$

$$
\begin{array}{lll}
a_i^T \cdot x \geq b_i & p_i \geq 0 & i \in M_1 \\
a_i^T x \leq b_i & p_i \leq 0 & i \in M_2 \\
a_i^T x = b_i & p_i \ \text{free} & i \in M_3 \\
x_j \geq 0 & p^T \cdot A_j \leq c_j & j \in N_1 \\
x_j \leq 0 & p^T \cdot A_j \geq c_j & j \in N_2 \\
x_j \ \text{free} & p^T \cdot A_j = c_j & j \in N_3
\end{array}
$$

__Theorem__: The dual of the dual LP is the primal LP.

__Theorem__: Let $\Pi_1$ and $\Pi_2$ be two LPs where $\Pi_2$ has been obtained from $\Pi_1$ by (several) transformations of the following type
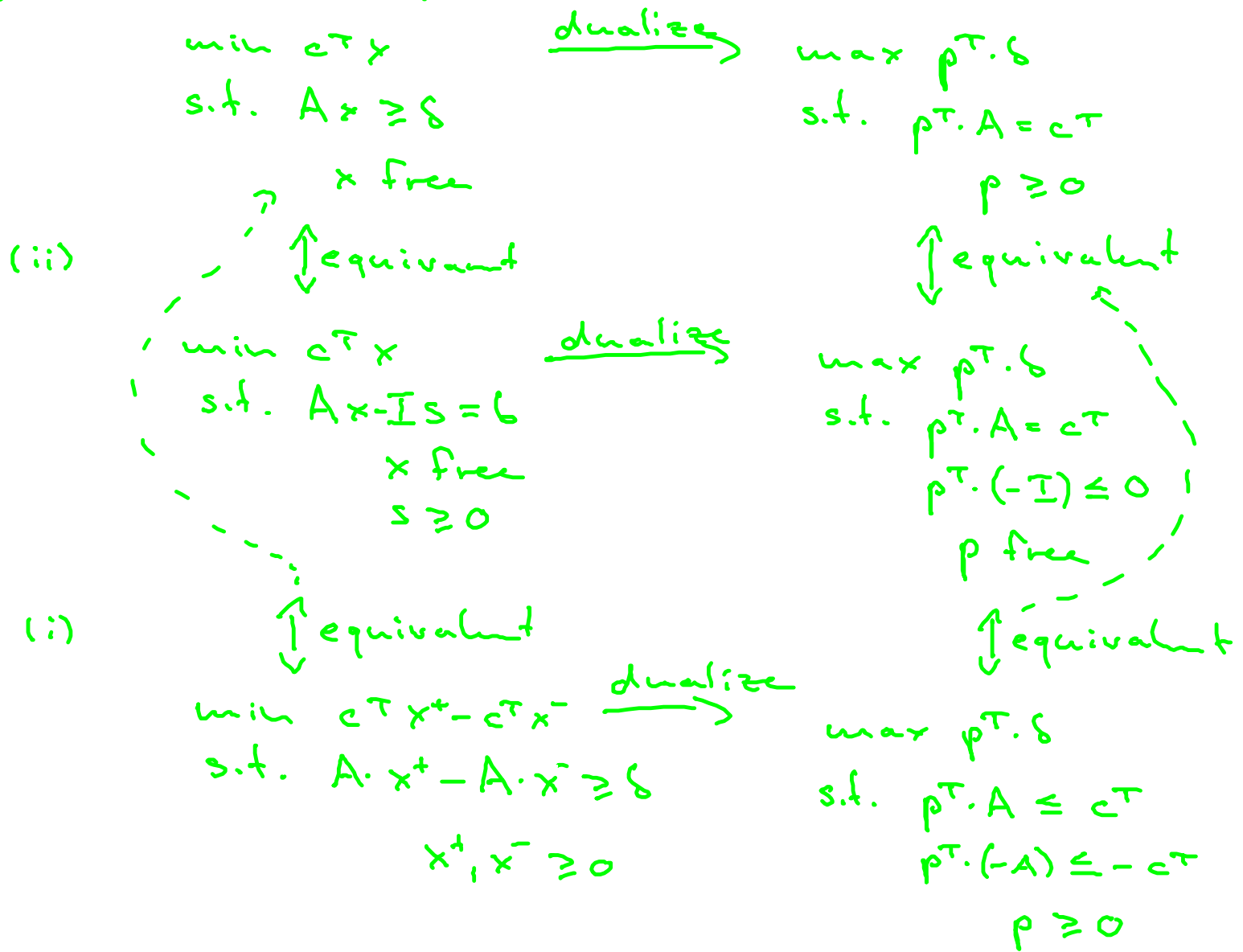
(i) Replace a free variable by difference of two non-negative variables.

(ii) Introducing a slack variable in order to replace an inequality constr. by equal. constr.

(iii) If some row of a feasible equality system is a linear combination of the other rows, eliminate this row.

Then the dual of $\Pi_1$ is equivalent to the dual of $\Pi_2$.

## Sketch of proof:

$$\min c^T x \quad \xrightarrow{\text{dualize}} \quad \max p^T \cdot b$$
$$\text{s.t. } Ax \geq b \qquad\qquad \text{s.t. } p^T \cdot A = c^T$$
$$x \text{ free} \qquad\qquad\qquad p \geq 0$$

(ii)

$\uparrow$ equivalent $\qquad\qquad\qquad\qquad \updownarrow$ equivalent

$$\min c^T x \quad \xrightarrow{\text{dualize}} \quad \max p^T \cdot b$$
$$\text{s.t. } Ax - Is = b \qquad\qquad \text{s.t. } p^T \cdot A = c^T$$
$$x \text{ free} \qquad\qquad\qquad p^T \cdot (-I) \leq 0$$
$$s \geq 0 \qquad\qquad\qquad p \text{ free}$$

(i)

$\updownarrow$ equivalent $\qquad\qquad\qquad\qquad \updownarrow$ equivalent

$$\min c^T x^+ - c^T x^- \xrightarrow{\text{dualize}} \quad \max p^T \cdot b$$
$$\text{s.t. } A \cdot x^+ - A \cdot x^- \geq b \qquad \text{s.t. } p^T \cdot A \leq c^T$$
$$x^+, x^- \geq 0 \qquad\qquad\qquad p^T \cdot (-A) \leq -c^T$$
$$p \geq 0$$

(iii) Let $a_1^T, \ldots, a_m^T$ rows of $A$ and suppose that

$$a_m^T = \sum_{i=1}^{m-1} \gamma_i \cdot a_i^T \quad \text{and} \quad b_m = \sum_{i=1}^{m-1} \gamma_i \cdot b_i$$

Then row $m$ is redundant and can be eliminated. The dual constraints

$$\sum_{i=1}^{m} p_i \cdot a_i^T \leq c^T \quad \text{can be rewritten}$$

as $\sum_{i=1}^{m-1} (p_i + \gamma_i \cdot p_m) \cdot a_i^T \leq c^T$

and the dual cost is

$$\sum_{i=1}^{m} p_i \cdot \delta_i = \sum_{i=1}^{m+n} (p_i + \gamma_i \cdot p_m) \cdot \delta_i$$

Let $q_i := p_i + \gamma_i \cdot p_m$

dual LP $\overset{equiv.}{\rightsquigarrow}$ $\max \sum_{i=1}^{m+n} q_i \cdot \delta_i$

$$\text{s.t. } \sum_{i=1}^{m+n} q_i \cdot a_i^T \leq c^T \qquad \square$$

# 4.3 Duality Theorem

<u>Theorem</u> (weak duality):

If $x$ feasible sol. to primal LP and

$p$ " " " dual " , then

$$c^T x \geq p^T \cdot \delta.$$

<u>Proof:</u> Consider a primal-dual pair of LPs.

$\min c^T x$ $\qquad$ $\max p^T \cdot \delta$

s.t. $Ax \geq b$ $\quad$ (1) $\qquad$ s.t. $p^T A \leq c^T$ $\quad$ (2)

$\qquad x \geq 0$ $\qquad\qquad\qquad\qquad p \geq 0$

$$c^T x \underset{\uparrow}{\geq} (p^T \cdot A) \cdot x = p^T \cdot (A \cdot x) \underset{\uparrow}{\geq} p^T \cdot \delta$$

$\qquad$ p fulfills (2) $\qquad\qquad\qquad$ x fulfills (1)

$\qquad$ x $\geq$ 0 $\qquad\qquad\qquad\qquad$ p $\geq$ 0 $\qquad\qquad\square$

<u>Corollary:</u> (i) If primal LP is unbounded (i.e. opt. cost $= -\infty$), then the dual LP is infeasible.

(ii) If dual LP is unbounded, then primal LP is infeasible.

**<u>Corollary</u>:** If $x, p$ are feasible sol. to the primal and dual LP, respectively, and if $c^T x = p^T b$, then $x$ and $p$ are optimal solutions.

**<u>Theorem (strong duality)</u>:**
If an LP has an optimal solution, so does its dual and the optimal costs are equal.

**<u>Proof</u>:** Consider LP in standard form with linearly indep. rows of $A$:

$$\min \; c^T x \qquad\qquad \max \; p^T b$$
$$\text{s.t.} \;\; Ax = b \qquad\qquad \text{s.t.} \;\; p^T A \le c^T$$
$$\qquad\quad x \ge 0$$

The simplex method yields opt. basis $B$ with
$$\bar{z}^T = c^T - c_B^T \cdot B^{-1} \cdot A \ge 0$$

and a corresp. opt. basic feas. sol. $x$ with
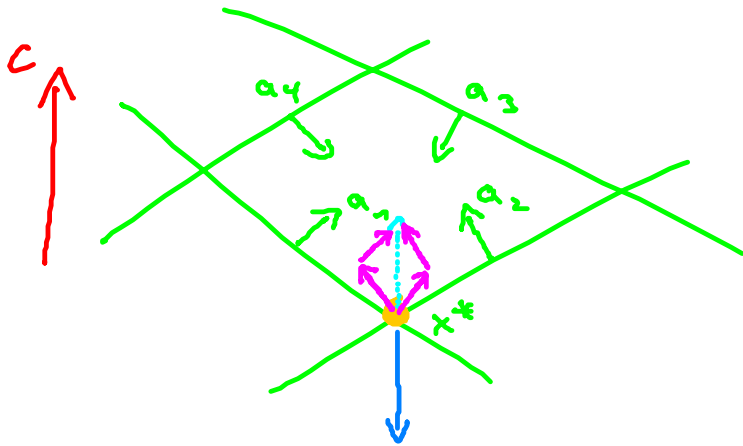$$x_B = B^{-1} \cdot b.$$

Let $p^T := c_B^T \cdot B^{-1}$. Then

$$p^T \cdot A = c_B^T \cdot B^{-1} \cdot A \le c^T$$

and $p^T \cdot b = c_B^T \cdot (B^{-1} \cdot b) = c_B^T \cdot x_B = c^T \cdot x.$ $\qquad \square$

**<u>Interpretation of duality</u>:**

$$\min c^T x$$
$$\text{s.t. } Ax \geq b$$

$$\max p^T b$$
$$\text{s.t. } p^T \cdot A = c^T$$
$$p \geq 0$$



$$c^T x^* = p^{*T} \cdot b$$

$$c^T \cdot x^* = p^{*T} \cdot A \cdot x^* \geq p^{*T} \cdot b$$

$$\Rightarrow p_i = 0 \text{ if } a_i^T \cdot x^* > b_i$$

## Complementary slackness:

**Theorem:** Let $x, p$ feasible primal and dual solutions, then $x, p$ are both optimal if and only if:

$$u_i := p_i \cdot (a_i^T x - b_i) = 0 \quad \forall i$$

$$v_j := (c_j - p^T \cdot A_j) \cdot x_j = 0 \quad \forall j$$

**Proof:** Feasibility $\Rightarrow u_i, v_j \geq 0 \quad \forall i, j$

$$c^T x \geq (p^T \cdot A) \cdot x = \underbrace{p^T \cdot (Ax)}_{(*)} \geq \underbrace{p^T b}_{(**)}$$

similar
for $v_j$'s.

$$(*) - (**) = \sum u_i$$

$\square$