

Weak duality:  $x, p$  feasible  $\Rightarrow c^T x \geq p^T b$

Strong duality: If primal LP has opt. sol., then also dual LP has opt. sol. and optimal costs are equal.

Complementary Slackness:  $x, p$  feasible

$$x, p \text{ both optimal} \Leftrightarrow \begin{aligned} p_i \cdot (a_i^T x - b_i) &= 0 \quad \forall i \\ x_j \cdot (c_j - p^T A_j) &= 0 \quad \forall j \end{aligned}$$

All different possibilities for primal-dual pair of LPs:

primal \ dual	finite opt.	unbounded	infeasible
finite opt.	✓	—	—
unbounded	—	—	✓
infeasible	—	✓	✓

Example where primal LP and dual LP are infeas.

$$\begin{aligned} \min \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 = 1 \\ & 2x_1 + 2x_2 = 3 \end{aligned}$$

$$\begin{aligned} \max \quad & p_1 + 3 \cdot p_2 \\ \text{s.t.} \quad & p_1 + 2p_2 = 1 \\ & p_1 + 2p_2 = 2 \end{aligned}$$

A geometric view

$$\begin{aligned} \min \quad & c^T x \\ \text{s.t.} \quad & Ax \geq b \end{aligned}$$

$$\begin{aligned} \max \quad & p^T b \\ \text{s.t.} \quad & p^T A = c^T \\ & p \geq 0 \end{aligned}$$

$$\begin{aligned} A &\in \mathbb{R}^{m \times n} \\ \text{rank}(A) &= n \end{aligned}$$

Let  $I \subseteq \{1, \dots, m\}$  with  $a_i, i \in I$ , linearly independent.  
 $|I| = n$

$\rightarrow a_i^T x = b_i, i \in I$ , has unique solution  $x^I$   
(basic solution)

Let  $p \in \mathbb{R}^m$  (dual vector).  $x^I, p$  are optimal if

(i)  $a_i^T \cdot x^I \geq b_i \quad \forall i$  (primal feasibility)

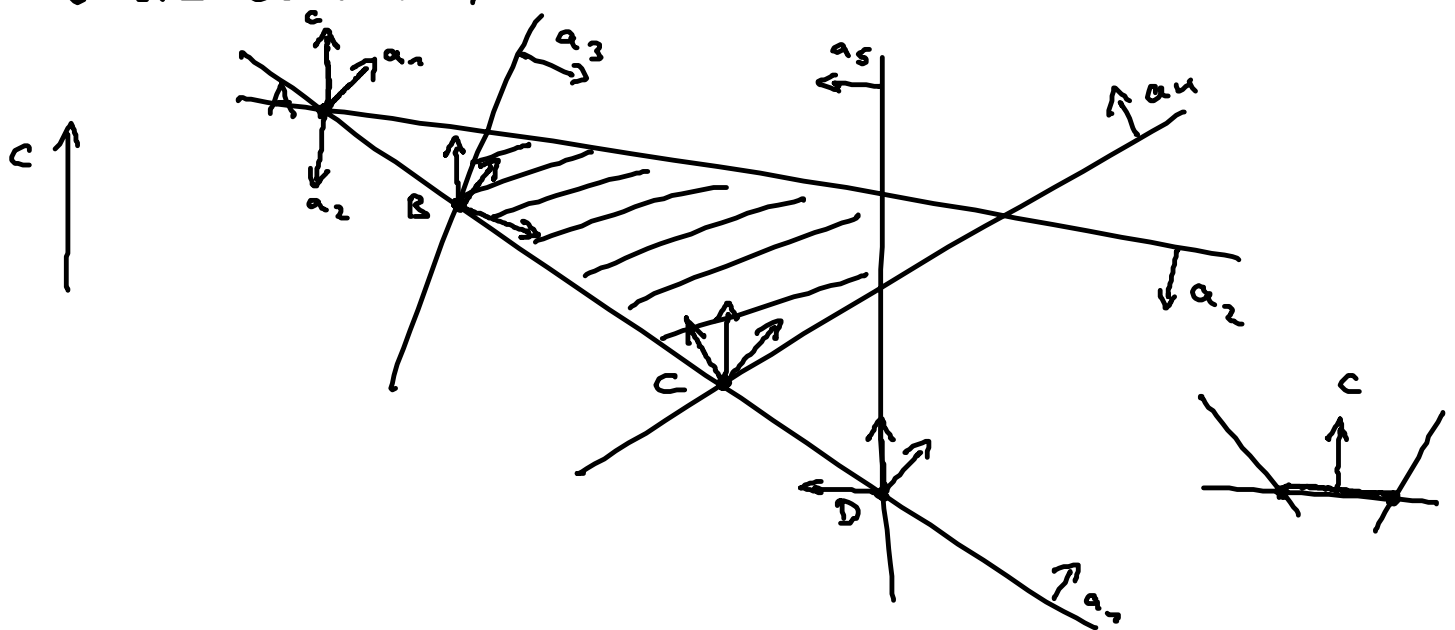
(ii)  $p_i = 0 \quad \forall i \notin I$  (compl. slackness)

(iii)  $\sum_{i=1}^m p_i \cdot a_i = c$  (dual feasibility)

(iv)  $p \geq 0$  ( " " )

(i) & (iii)  $\Rightarrow \sum_{i \in I} p_i \cdot a_i = c \rightarrow$  unique solution  $p^I$

( $a_i, i \in I$ , form basis for dual problem,  $p^I$  is corresp. basic solution)



A is primal and dual infeasible

B is primal feasible, dual infeasible

C is primal feasible, dual feasible

D is primal infeasible, dual feasible.

#### 4.4 Optimal dual variables as marginal costs

$$\begin{aligned} \min c^T x \\ \text{s.t. } Ax = b \\ x \geq 0 \end{aligned}$$

$$\begin{aligned} \max p^T b \\ \text{s.t. } p^T A \leq c^T \end{aligned}$$

Let  $x^*$  be optimal basic feas. sol. for primal LP with basis  $B$ , i.e.  $x_B^* = B^{-1} \cdot b$ . Assume that  $x^*$  non-deg.,  $x_B^* > 0$ . Replace  $b$  by  $b+d$ . For small  $d$ ,  $B$  remains optimal basis, since:

$$B^{-1} \cdot (b+d) = B^{-1} \cdot b + B^{-1} \cdot d \geq 0 \quad (\text{feasibility})$$

$$\text{and } \bar{c}^T = c^T - c_B^T \cdot B^{-1} \cdot A \geq 0 \quad (\text{optimality})$$

Optimal cost of the perturbed problem is:

$$c_B^T \cdot B^{-1} \cdot (b+d) = c_B^T \cdot x_B^* + \underbrace{(c_B^T \cdot B^{-1})}_{p} \cdot d$$

$\rightarrow p_i$  is marginal cost per unit increase of  $b_i$ .

#### 4.5 The dual simplex method

$$\begin{aligned} \min c^T x \\ \text{s.t. } Ax = b \\ x \geq 0 \end{aligned}$$

$$\begin{aligned} \max p^T b \\ \text{s.t. } p^T A \leq c^T \end{aligned}$$

basis  $B \rightarrow$  dual variables  $p^T = c_B^T \cdot B^{-1}$   
 primal opt. cond.  $\leftrightarrow$  dual feasibility

$$\begin{array}{c|ccc} -c_B^T x_B & \bar{c}_1 & \dots & \bar{c}_n \\ \hline x_{B(1)} & | & & | \\ \vdots & B^{-1} A_{1j} & \dots & B^{-1} A_{1n} \\ x_{B(m)} & | & & | \end{array}$$

Assume that  $\bar{c}_i \geq 0 \forall i$ , i.e.  $p^T A = c_B^T \cdot B^{-1} \cdot A \leq c^T$   
 and  $p$  is feasible with cost

$$p^T \cdot b = c_B^T \cdot B^{-1} \cdot b = c_B^T \cdot x_B = c^T \cdot x$$

If  $B^{-1} \cdot b \geq 0$ , then  $x, p$  are optimal.

Assume that  $x_{B(l)} < 0$  and consider  $l$ -th row

$$(x_{B(l)} \quad v_1 \quad \dots \quad v_n) \quad (\text{pivot row})$$

I: Let  $j \in \{1, \dots, n\}$  with  $v_j < 0$  and

$$\frac{\bar{c}_j}{|v_j|} = \min_{\{i | v_i < 0\}} \frac{\bar{c}_i}{|v_i|}$$

Then, choosing  $v_j$  as the pivot element and performing one iteration of the simplex method yields a new basis  $B'$  with  $c_B^T \cdot B'^{-1} \cdot A \leq c^T$  and  $p'^T \cdot b \geq p^T \cdot b$  (with " $>$ " if  $\bar{c}_j > 0$ ).

II: If  $v_i \geq 0 \quad \forall i \in \{1, \dots, n\}$ , then the dual LP is unbounded and the primal LP is infeasible.

→ dual simplex method.

Remark: Dual simplex method terminates if a lexicographic pivoting rule is applied.

When to apply the dual simplex method?

If basic feasible solution of dual LP is readily available (but not for primal LP).

Example: We solved primal LP and only afterwards we realize that the right-hand side vector  $b$  is not correct → use iterations of dual

simplex method to reoptimize for correct r.h.s.  
(much faster than resolving LP from scratch.)

Example:

$$\min x_1 + x_2$$

$$\text{s.t. } x_1 + 2x_2 - x_3 = 2$$

$$x_1 - x_4 = 1$$

$$x_i \geq 0 \quad i=1, \dots, 4$$